Involution for the representations of Hecke algebras

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Motivation: the sign representation as an alternating sum

Let (W, S) be a finite Coxeter system. The ***sign representation*** ε of W is the representation defined by

$$\varepsilon(s) = -1 \text{ for all } s \in S.$$

The map ε extends naturally to a linear representation of W. In case (W, S) is realized as a finite group generated by reflections associated with a root system in a euclidean space, we have

$$\varepsilon(w) = \det w \text{ for all } w \in W.$$

One can prove that

$$\varepsilon = \sum_{J \subset S} (-1)^{|J|} \left(1_{W_J}\right)^W$$

where $\left(1_{W_J}\right)^W$ denotes the trivial character of W_J induced to W.

History

- **(L. Solomon)** For any character χ of a **finite Weyl group** (W, S), there is a linear map D_W defined as follows:
$$D_W(\chi) := \sum_{I \subset S} (-1)^{|I|} \operatorname{Ind}_{W_I}^W \operatorname{Res}_{W_I}^W(\chi) = \chi \otimes \operatorname{sgn}. \tag{1}$$
- **(Alvis-Curtis-Kawanaka)** Let $G/\mathbb{F}_q$ be a **finite group of Lie type** satisfying some assumptions. There is a linear map on $K_0(\mathbb{C}G)$ defined by
$$D_G = \sum_{I \subset S} (-1)^{|I|} R_{L_I}^G \circ {}^*R_{L_I}^G$$
where L_I is the standard Levi complement of the standard parabolic subgroup P_I.
- **(S. Kato [6])** There is a generalization of L. Solomon's result for modules of **equal-parameter finite/affine Hecke algebras**:
$$D_H[M] := \sum_{I \subset S} (-1)^{|I|} [\operatorname{Ind}_I(\operatorname{Res}_I M)] = [M^*]. \tag{2}$$
where the induction functor Ind_I is defined by $\operatorname{Ind}_I N := \mathcal{H} \otimes_{\mathcal{H}_I} N$ for an $\mathcal{H}_I$-module N, and M^* is an analogue of the right hand side of (1) for the Hecke algebra modules.
- **(Aubert-Zelevinsky)** Let G be a **connected p-adic reductive group**. For any complex smooth finite length representation π of G, one can define the map on its Grothendieck group:
$$D_G([\pi]) = [\sum_{I \subset S} (-1)^{|I|} \mathrm{i}_{P_I}^G \circ \mathrm{r}_{P_I}^G(\pi)] \tag{3}$$
where $\mathrm{i}_{P_I}^G$ and $\mathrm{r}_{P_I}^G$ are normalized induction and normalized Jacquet modules.

What is a Hecke algebra?

For a (finite) Coxeter system (W, S) and parameters $q_s \in \mathbb{C}^\times \backslash \{\text{roots of unity}\}$, define the Hecke algebra $H = H(W, q_s)$ as the $\mathbb{C}$-algebra with basis T_w, $w \in W$ satisfying:

$$(T_s + 1)(T_s - q_s) = 0, \text{ for } s \in S, \tag{4}$$

$$T_w \cdot T_{w'} = T_{ww'}, \text{ if } \ell(w) + \ell(w') = \ell(ww'). \tag{5}$$

Tip: When we refer to something as a "Hecke algebra", we are generally describing an algebra with generator relations of a form similar to this, though it can sometimes be more complicated. We use $\mathcal{H}$ to denote an affine Hecke algebra.

Our ultimate goal

Find the involution formula for general Hecke algebras, namely those corresponding to an arbitrary block $\operatorname{Rep}^{\mathfrak{s}}(G)$.

Some guiding philosophy

- The Hecke algebra is the "deformed group algebra" of some Weyl group, as an example, we may recover (1) from (2) by requiring $q_s = 1$. On the other hand, once we have a result for Weyl groups, we expect to have an analogue for Hecke algebras.
- The results for finite groups of Lie type have analogues for p-adic groups in the depth-zero case, although these analogues can be much more complicated.

We therefore anticipate the existence of an involution formula for the depth-zero case!

Involution for affine Hecke algebras

Main Theorem: Define a twisted action $*$ on $\mathcal{H}$ as follows: set

$$T_w^* = (-1)^{l(w_{\text{fin}})} q(w) T_{w^{-1}}^{-1}$$

where $q(w) = \prod_{k=1}^{r} q_{s_{i_k}}$. Given $\mathcal{H}$-module (π, M), let (π^*, M^*) be the $\mathcal{H}$-module such that $M^* = M$ as K-vector space, equipped with the $\mathcal{H}$-action $\pi^*(h)(m) := \pi(h^*)(m)$, $\forall m \in M$ and $\forall h \in \mathcal{H}$. Then we have the following equality in the Grothendieck group:

$$D[M] = [M^*].$$

Our Results Obtained So Far (in words)

- We find a generalization of S-I. Kato's results (2) to the unequal parameter case, furthermore along the way we add more details to the proofs and correct the errors.
- In Howlett-Lehrer's articles, they find a generalization of (1) to arbitrary characters of some normalizer subgroup. We find an analogue of Howlett-Lehrer's duality formula for the Hecke algebras under some assumptions.
- We restrict the Alvis-Curtis-Kawanaka duality to a fixed Harish-Chandra series, via (6) we obtain the left hand side of involution for these Hecke algebras which explains the above result obtained from Howlett-Lehrer's theory.
- (p-adic case) Inspired by the finite Hecke algebra case, we obtain the left hand side of involution for the affine Hecke algebra modules, where the Hecke algebra is associated to an arbitrary Bernstein block.
- We study the Aubert-Zelevinsky duality on the Bernstein blocks of the exceptional group G_2 and the corresponding involution for Hecke algebras based on the recent works of Aubert-Xu in [3] and [4].

How does a Hecke algebra arise from group representations ?

The finite group case: Let $\mathbf{G}$ be a connected reductive algebraic group defined over $\mathbb{F}_q$, and G is its rational points. The *Harish-Chandra series* corresponding to a cuspidal pair (L, Λ) is defined as

$$\operatorname{Irr}_{\mathbb{C}}(G|(L, \Lambda)) = \left\{ M \in \operatorname{Irr} G \mid R_L^G(\Lambda) \twoheadrightarrow M \right\} / \sim .$$

- The set of irreducible representations of G is partitioned into Harish-Chandra series:
$$\operatorname{Irr}_{\mathbb{C}} G = \bigsqcup_{(L, \Lambda) \in \mathcal{C}(G)/\sim} \operatorname{Irr}_{\mathbb{C}} (G \mid (L, \Lambda))$$
where $\mathcal{C}(G)$ denotes the set of cuspidal pairs of G.
- There is a bijection between
$$\operatorname{Irr}_{\mathbb{C}}(G|(L, \Lambda)) \leftrightarrow \operatorname{Irr}_{\mathbb{C}}(\operatorname{End}_G(R_L^G \Lambda)). \tag{6}$$
induced by the Hom-functor $\mathcal{E}_G : M \mapsto \operatorname{Hom}_G(R_L^G(\Lambda), M)$.

The p-adic case: The category $\operatorname{Rep}(G(F))$ of smooth complex representations of $G(F)$ has a *Bernstein decomposition* as a product of subcategories

$$\prod_{\mathfrak{s} \in \mathfrak{B}(G(F))} \operatorname{Rep}^{\mathfrak{s}}(G(F)),$$

parametrized by inertial supports $\mathfrak{s} = [L, \sigma]_G$. There are two approaches to study $\operatorname{Rep}^{\mathfrak{s}}(G(F))$:

- The approach via the theory of types, developed by C.J. Bushnell & P.C. Kutzko [5], A. Roche, S. Stevens (among others):
$$\operatorname{Rep}^{\mathfrak{s}}(G(F)) \cong \operatorname{Mod-End}_{G(F)} \left(\text{c-ind}_K^{G(F)}(\rho) \right).$$
- The approach by J. Bernstein, A. Roche, M. Solleveld (among others):
$$\operatorname{Rep}^{\mathfrak{s}}(G(F)) \cong \operatorname{Mod-End}_{G(F)} \left(I_P^G \left(\text{c-ind}_{L^1}^{L(F)} (\sigma_1) \right) \right).$$

Good news:

- The above two endomorphism algebras $\operatorname{End}_G(R_L^G \Lambda)$ and $\operatorname{End}_{G(F)} \left(I_P^G \left(\text{c-ind}_{L^1}^{L(F)} (\sigma_1) \right) \right)$ are Hecke algebras and their generator relations can be described explicitly.
- The involution for Weyl groups (1) or Hecke algebras (2) can be seen via twisting (see the right-hand sides) on elements of the Hecke algebras.

Hecke algebra side vs Group side

Commutative diagrams for the finite (and the p-adic) cases:

$$\begin{array}{ccc} \mathbb{Z}\operatorname{Irr}_{\mathbb{C}}(G|(L_0, \Lambda)) \;/\; \operatorname{Rep}^{\mathfrak{s}}(G) & \xrightarrow{\mathcal{E}_G} & \mathbb{Z}\operatorname{Irr}_{\mathbb{C}}(\operatorname{End}_G(R_{L_0}^G \Lambda)) \;/\; \operatorname{Mod-End}_G(\mathrm{i}_P^G \Sigma) \\ \downarrow {\scriptstyle \operatorname{pr}_{(-)} \circ \text{``Res''}_{(-)}^{(-)}} & & \downarrow {\scriptstyle \operatorname{Res}_{(-)}^{(-)}} \\ \operatorname{Rep}^{\mathfrak{s}_{L_I}}(L_I) \;/\; \mathbb{Z}\operatorname{Irr}_{\mathbb{C}}(L_I|({}^wL_0, {}^w\Lambda)) & \xrightarrow{\mathcal{E}_{(-)}} & \operatorname{Mod-End}_{L_I}(\mathrm{i}_{L_I \cap P}^{L_I} \Sigma_w) \;/\; \mathbb{Z}\operatorname{Irr}_{\mathbb{C}}(\operatorname{End}_{L_I}(R_{{}^wL_0}^{L_I} {}^w\Lambda)) \end{array} \tag{7}$$

$$\begin{array}{ccc} \mathbb{Z}\operatorname{Irr}_{\mathbb{C}}(G|(L_0, \Lambda)) \;/\; \operatorname{Rep}^{\mathfrak{s}}(G) & \xrightarrow{\mathcal{E}_G} & \mathbb{Z}\operatorname{Irr}_{\mathbb{C}}(\operatorname{End}_G(R_{L_0}^G \Lambda)) \;/\; \operatorname{Mod-End}_G(\mathrm{i}_P^G \Sigma) \\ \uparrow {\scriptstyle \text{``Ind''}_{(-)}^{(-)}} & & \uparrow {\scriptstyle \operatorname{Ind}_{(-)}^{(-)}} \\ \operatorname{Rep}^{\mathfrak{s}_{L_I}}(L_I) \;/\; \mathbb{Z}\operatorname{Irr}_{\mathbb{C}}(L_I|({}^wL_0, {}^w\Lambda)) & \xrightarrow{\mathcal{E}_{(-)}} & \operatorname{Mod-End}_{L_I}(\mathrm{i}_{L_I \cap P}^{L_I} \Sigma_w) \;/\; \mathbb{Z}\operatorname{Irr}_{\mathbb{C}}(\operatorname{End}_{L_I}(R_{{}^wL_0}^{L_I} {}^w\Lambda)) \end{array} \tag{8}$$

We find the left hand side of the involution formula:

(The finite group case)

$$\mathrm{D}_H = \sum_{I \subset S} (-1)^{|I|} \Big(\sum_{w \in \mathfrak{S}_I} \operatorname{Ind}_{\operatorname{End}_{L_I}(R_{{}^wL_0}^{L_I} {}^w\Lambda)}^{\operatorname{End}_G(R_{L_0}^G \Lambda)} \circ \operatorname{Res}_{\operatorname{End}_{L_I}(R_{{}^wL_0}^{L_I} {}^w\Lambda)}^{\operatorname{End}_G(R_{L_0}^G \Lambda)} \Big). \tag{9}$$

(The p-adic case)

$$\mathrm{D}_{\mathcal{H}} = \sum_{I \subset S} (-1)^{|I|} \Big(\sum_{\mathfrak{s}_{L_I} \in \mathfrak{S}_I} \operatorname{Ind}_{\operatorname{End}_{L_I}(\mathrm{i}_{L_I \cap P}^{L_I} \Sigma_w)}^{\operatorname{End}_G(\mathrm{i}_P^G \Sigma)} \circ \operatorname{Res}_{\operatorname{End}_{L_I}(\mathrm{i}_{L_I \cap P}^{L_I} \Sigma_w)}^{\operatorname{End}_G(\mathrm{i}_P^G \Sigma)} \Big). \tag{10}$$

Moreover, for **the finite group case**, under some assumptions, let M^* denote the module M endowed with the twisted action of $\operatorname{End}_G(\operatorname{Ind}_{P_{I_0}}^G(\Lambda))$ defined by

$$T_w^* = (-1)^{|I_0| + \ell_{I_0^\perp}(w)} p_w T_{w^{-1}}^{-1}.$$

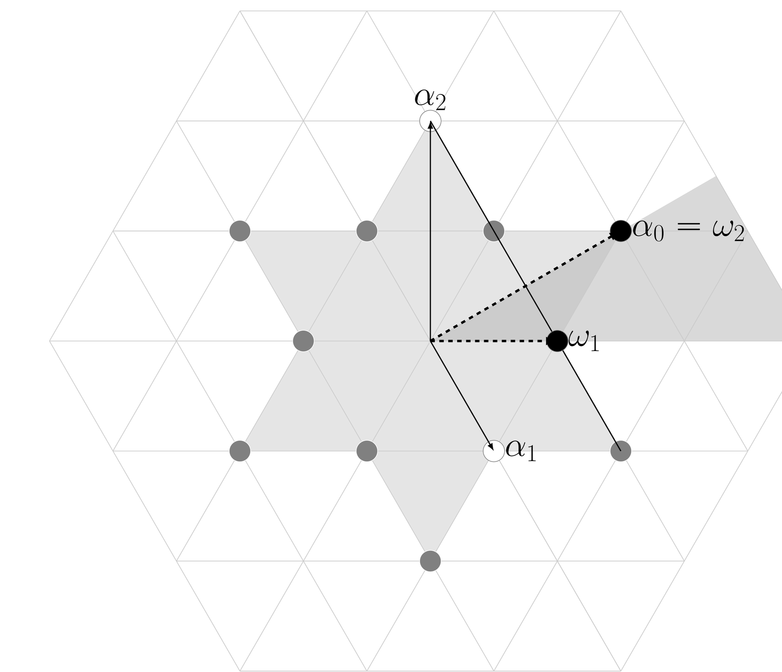
Then we have the following equality in the Grothendieck group of $\operatorname{End}_G(\operatorname{Ind}_{P_{I_0}}^G(\Lambda))$-modules:

$$\sum_{I \subset S} (-1)^{|I|} \sum_{w \in W_I \backslash C_{I_0}(I)/W(\Lambda)} [\operatorname{Ind}_{E_I'}^{E_G(\Lambda)} [\operatorname{Res}_{E_I'}^{E_G(\Lambda)}(M)]] = [M^*], \tag{11}$$

where E_I' is the subalgebra of $E_G(\Lambda)$ spanned by $\{T_v \mid v \in W(\Lambda) \cap W_I^w\}$.

An example: p-adic group G_2

Root/weight lattice diagram of G_2:



An affine Weyl group of this type is denoted by $W_{\widetilde{G_2}} = W(\mathcal{R}^\vee)$ with the set $S_{\text{aff}} = \{s_1, s_2, s_0 = t_{\alpha_0} s_2 s_1 s_2 s_1 s_2\}$ where $s_i = s_{\alpha_i}$, $i = 1, 2$. Denote by $\mathcal{H} = \mathcal{H}(W_{\widetilde{G_2}}, q_s)$ the affine Hecke algebra.

- We can check directly that the involution interchanges the Steinberg module and the trivial module.
- Using the method employed by G. Muic and based on the recent work of Aubert-Xu, we are able to compute the A-Z duality for each block and, consequently, determine the involution on the Hecke algebra side.

Further research

- Remove the assumption we made while proving (11).
- Find the right hand side of the involution for the depth-zero case.
- Try to find formula for the general case via the forthcoming work of Adler-Fintzen-Mishra-Ohara ([1] and [2]).

Type theory is very important to us!

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