1. Elementary matrices and calculating determinant via Gauss-Jordan elimination

Definition 1.1. An $n \times n$ matrix E is an elementary matrix if it can be obtained by performing a single elementary row operation on the identity matrix I_n.

See Exercise 82-90 of [1] or [2] and [3].

Some examples of elementary matrices for n = 3 and for each of the elementary row operations are:

Row-switching transformations (determinant is -1): $E_{R_1 \leftrightarrow R_3} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, which is obtained by interchanging the first row and third rows of I_3.

Similarly, we have

$$E_{R_1 \leftrightarrow R_2} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{R_2 \leftrightarrow R_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

which are obtained by interchanging the first row and second rows of I_3 (*resp.* the second row and third rows of I_3).

Row-multiplying transformations (determinant is a): $E_{aR_1} = \begin{bmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ which is obtained by multiplying the first row of I_3 by a , for any $a \in \mathbb{R}$.

Similarly, we have

$$E_{aR_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{aR_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{bmatrix}$$

which is obtained by multiplying the second row (*resp.* the third row) of I_3 by a, for any $a \in \mathbb{R}$.

Row-addition transformations (determinant is 1): $E_{R_2 \leftarrow R_2 + aR_1} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ which is obtained by multiplying the first row of I_3 by a and adding it to the second row of I_3.

Similarly, we have

$$E_{R_3 \leftarrow R_3 + aR_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & 0 & 1 \end{bmatrix}, \quad E_{R_3 \leftarrow R_3 + aR_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a & 1 \end{bmatrix}$$

which is obtained by multiplying the first row (*resp.* the second row) of I_3 by a and adding it to the third row of I_3.

Proposition 1.2. *Left multiplication of an elementary matrix realizes the corresponding row operation.*

Example 1.3. Consider

$$E_{R_1 \leftrightarrow R_2} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} a & b \\ g & d \\ e & f \end{bmatrix}$$

then we have

$$E_{R_1 \leftrightarrow R_2} A = \begin{bmatrix} g & d \\ a & b \\ e & f \end{bmatrix}$$

We see left multiplication by $E_{R_1 \leftrightarrow R_2}$ does the work of interchanging R_1 and R_2 of A!

Remark. These elementary matrices are often denoted by E_i, for $i \in \mathbb{N}$. The subscripts on E have no particular meaning but are just used to distinguish one elementary matrix from the next. We keep using our notations.

We now explain how to find the determinant via Gauss-Jordan elimination.

Consider the matrix $A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -6 & 2 \\ -1 & 3 & 0 \end{bmatrix}$. This can be done by multiplying A on the left by the elementary matrix $E_{R_1 \leftrightarrow R_3}$ to yield

$$E_{R_1 \leftrightarrow R_3} A = \begin{bmatrix} -1 & 3 & 0 \\ 2 & -6 & 2 \\ 0 & 1 & -1 \end{bmatrix}.$$

Suppose we then wish to perform the elementary row operation of multiplying the first row by -1 , we can do this by multiplying by E_{-R_1} which then gives

$$E_{-R_1} E_{R_1 \leftrightarrow R_3} A = \begin{bmatrix} 1 & -3 & 0 \\ 2 & -6 & 2 \\ 0 & 1 & -1 \end{bmatrix}.$$

We can then perform on this result the elementary row operation of multiplying the first row by -2 and adding it to the second row using $E_{R_2 \leftarrow R_2 - 2R_1}$ to obtain

$$E_{R_2 \leftarrow R_2 - 2R_1} E_{-R_1} E_{R_1 \leftrightarrow R_3} = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}.$$

If our goal is to obtain rref for A, then we would continue with the following elementary matrices to switch rows 2 and 3 we apply $E_{R_2 \leftrightarrow R_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ to the previous result we get

$$E_{R_2 \leftrightarrow R_3} E_{R_2 \leftarrow R_2 - 2R_1} E_{-R_1} E_{R_1 \leftrightarrow R_3} A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Taking determinants on both sides, we have the determinants equation as follows (recall that $\det(AB) = \det(A)\det(B)$):

$$\begin{aligned} (-1) \times 1 \times (-1) \times (-1) \det(A) &= \det(E_{R_2 \leftrightarrow R_3}) \det(E_{R_2 \leftarrow R_2 - 2R_1}) \det(E_{-R_1}) \det(E_{R_1 \leftrightarrow R_3}) \det(A) \\ &= \det\left(\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \right) = 2. \end{aligned}$$

Therefore $\det(A) = -2$.

References

[1] Otto Bretscher, *Linear Algebra with Application*, 5th ed., Pearson, December 20, 2012.

[2] Wikipedia contributors. *Elementary matrix.* Wikipedia, The Free Encyclopedia. https://en.wikipedia.org/wiki/Elementary_matrix

[3] Jesús De Loera. *Handout on Elementary Matrices*. UC Davis Mathematics.
https://www.math.ucdavis.edu/~deloera/TEACHING/LOWDIVISION/MATH22A/handoutelemmat.pdf