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Involution pour les représentations des algèbres de Hecke

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業精於勤，荒於嬉。

Academic excellence is achieved through diligence, while studies are neglected and deteriorate when one frolics

韓愈 «進學解»

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Résumé

Dans cette thèse, nous présentons deux généralisations de la dualité d'Alvis-Curtis au cas des algèbres de Hecke : une version relative pour les algèbres de Hecke finies, basée sur le travail de Howlett-Lehrer, et une version à paramètres inégaux pour les algèbres de Hecke affines, basée sur le travail de S.-I. Kato (qui, sous certaines hypothèses, correspond à la dualité d'Aubert-Zelevinsky pour les représentations irréductibles lisses complexes de groupes p-adiques). Nous démontrons ensuite leur compatibilité avec la dualité d'Aubert-Zelevinsky lorsqu'elles sont restreintes à certains blocs de Bernstein. Enfin, motivés par le travail récent d'Aubert-Xu, nous fournissons des exemples de calculs du foncteur de dualité pour les séries principales du groupe exceptionnel G_2.

Mots-clés

Dualité d'Aubert-Zelevinsky, Dualité d'Alvis-Curtis, Involution de Kato, algèbres de Hecke, représentations des groupes p-adiques

Abstract

In this thesis, we give two generalizations of the Alvis-Curtis duality for Hecke algebras: a relative version for finite Hecke algebras, based on Howlett-Lehrer's work, and an unequal parameter version for affine Hecke algebras, based on S-I. Kato's work (which under certain assumptions, corresponds to the Aubert-Zelevinsky duality for complex smooth irreducible representations of p-adic groups). Then, we prove their compatibility with the Aubert-Zelevinsky duality when restricted to some Bernstein blocks. Finally, motivated by the recent work of Aubert-Xu, we provide examples of calculations of the duality functor for the principal series of the exceptional group G_2.

Keywords

Aubert-Zelevinsky duality, Alvis-Curtis duality, Kato's involution, Hecke algebras, representations of p-adic groups

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Chapter 1

Introduction en Français

1.1 Aperçu de l'histoire

L'étude de la dualité pour les représentations à coefficients complexes des groupes de Weyl finis et des groupes de type de Lie peut être retracée jusqu'aux années 1960, avec des contributions issues des travaux de D. Alvis [4], L. Solomon [58], et s'étendant à C.W. Curtis [25], R.B. Howlett & G.I. Lehrer [34] ainsi que P. Deligne & G. Lusztig [27] dans les années 1980. Plusieurs années plus tard, influencé par la démonstration de [27], S-I. Kato [39] a défini un opérateur de dualité pour les représentations des algèbres de Hecke associées à un groupe de Weyl fini, ou affine avec des paramètres égaux, et il a exprimé cet opérateur en termes d'une certaine involution sur l'algèbre de Hecke. Inspirée par le cas de l'algèbre de Hecke, A-M. Aubert a défini en 1995 [7] une dualité (aujourd'hui appelée la dualité d'Aubert-Zelevinsky) sur le groupe de Grothendieck de la catégorie des représentations lisses, de longueur finie, complexes d'un groupe p-adique, et elle a démontré plusieurs propriétés de cette dualité.

Pour être plus précis, introduisons davantage de notations. Soit F un corps local non-archimédien et G un groupe réductif connexe sur F. La catégorie $\mathrm{Rep}(G(F))$ des représentations complexes lisses de $G(F)$ possède une décomposition de Bernstein en produits de sous-catégories (appelées *blocs de Bernstein*)

$$\prod_{\mathfrak{s}\in\mathfrak{B}(G(F))} \mathrm{Rep}^{\mathfrak{s}}(G(F)),$$

paramétrées par les supports inertiaux $\mathfrak{s} = [L, \sigma]_G$ où L est un sous-groupe de Levi de G et σ est une représentation supercuspidale irréductible de $L(F)$.

Rappelons d'abord l'approche adoptée par J. Bernstein [13], A. Roche [52], M. Solleveld [57] (entre autres) pour étudier $\mathrm{Rep}^{\mathfrak{s}}(G(F))$. Choisissons un sous-groupe parabolique P tel que $P = LU$. Soit σ_1 une sous-représentation irréductible de $\sigma|_{L^1}$ où L^1 désigne l'intersection des noyaux des caractères non ramifiés de $L(F)$. Alors

$$\mathrm{Rep}^{\mathfrak{s}}(G(F)) \cong \mathrm{Mod}\text{-}\mathrm{End}_{G(F)}\left(I_P^G\left(\text{c-ind}_{L^1}^{L(F)}(\sigma_1)\right)\right),$$

où I_P^G désigne l'induction parabolique normalisée de $P(F)$ à $G(F)$ et $\text{c-ind}_{L^1}^{L(F)}(\sigma_1)$ est le petit progénerateur. D'après le travail de M. Solleveld [57, Théorème 10.9], l'algèbre d'endomorphismes $\mathrm{End}_{G(F)}\left(I_P^G\left(\text{c-ind}_{L^1}^{L(F)}(\sigma_1)\right)\right)$ est une extension d'une algèbre de Hecke affine par une algèbre de groupe tordue par un 2-cocycle.

Une autre approche pour étudier $\mathrm{Rep}^{\mathfrak{s}}(G(F))$ est la théorie des types, développée par C.J. Bushnell & P.C. Kutzko [19], A. Roche [31], S. Stevens [59] (entre autres). Sous certaines hypothèses, on sait qu'il existe une équivalence de catégories

$$\mathrm{Rep}^{\mathfrak{s}}(G(F)) \cong \mathrm{Mod}\text{-}\mathrm{End}_{G(F)}\left(\text{c-ind}_K^{G(F)}(\rho)\right),$$

si (K,ρ) est un $\mathfrak{s}$-type. Pour un type (K,ρ) de profondeur zéro, L. Morris a montré dans [46] que l'algèbre des endomorphismes $\mathrm{End}_{G(F)}\left(\text{c-ind}_K^{G(F)}(\rho)\right)$ est isomorphe à une extension d'une algèbre de Hecke affine par une algèbre de groupe tordue par un 2-cocycle.

De plus, l'involution de S-I. Kato peut être vue, sous certaines hypothèses, comme l'équivalent de la dualité Aubert-Zelevinsky sur la catégorie des modules d'une certaine algèbre de Hecke. Notre but est de d'établir la formule pour l'involution sur des algèbres de Hecke plus générales, notamment celles correspondant à un bloc arbitraire $\mathrm{Rep}^{\mathfrak{s}}(G)$.

1.2 Principaux résultats de cette thèse

Cette thèse est divisée en trois parties. La première partie généralise les résultats de S-I. Kato au cas des paramètres inégaux ; en outre, nous apportons des détails supplémentaires aux démonstrations et corrigeons les erreurs rencontrées en chemin. La deuxième partie commence par le théorème d'involution pour les algèbres de Hecke finies apparaissant dans la théorie de Howlett-Lehrer, qui sert de base à l'étude de l'involution pour un bloc de Bernstein arbitraire. Inspirés par le cas des algèbres de Hecke finies, nous obtenons le côté gauche de l'involution pour un bloc de Bernstein arbitraire. Dans la troisième partie, nous avons étudié la dualité d'Aubert-Zelevinsky sur les séries principales du groupe exceptionnel G_2 et l'involution correspondante pour les algèbres de Hecke, en nous basant sur les travaux récents d'Aubert-Xu dans [9] et [10].

Partie I : Involution pour les algèbres de Hecke affines généralisées avec paramètres inégaux

Nous présentons maintenant une introduction plus détaillée pour la première partie. Soit $\mathcal{R} := (X, R, Y, R^\vee)$ un système de racines satisfaisant aux axiomes énumérés dans [56, Section 1.3], et Δ désigne un ensemble de racines simples de R. Nous définissons s_α comme la réflexion par rapport à l'hyperplan orthogonal à α. Soit W un groupe de Weyl fini engendré par l'ensemble des réflexions simples $S = \{s_\alpha : \alpha \in \Delta\}$. Pour tout élément $w \in W$, nous introduisons sa longueur $\ell(w)$ comme étant le plus petit nombre tel que w puisse être écrit comme un produit de $\ell(w)$ réflexions simples, et $\mathrm{sgn}(w) := (-1)^{\ell(w)}$. Soit w_0 l'élément le plus long de W. Pour $I \subset S$ un sous-ensemble de S, soit $|I|$ le cardinal de I. Soit W_I le sous-groupe de W engendré par I. De plus, soit $\mathrm{Res}^W_{W_I}$ (resp. $\mathrm{Ind}^W_{W_I}$) le foncteur de restriction (resp. d'induction) pour les représentations des groupes finis W et W_I. Dans [58, Théorème 2], L. Solomon démontre que pour toute caractère χ de W, on a

$$\sum_{I\subset S} (-1)^{|I|} \mathrm{Ind}^W_{W_I} \mathrm{Res}^W_{W_I}(\chi) = \widehat{\chi} := \chi \otimes \mathrm{sgn}\,. \tag{1.2.1}$$

Du point de vue où les algèbres de Hecke peuvent être considérées comme des algèbres de groupe "déformées" associées aux groupes de Weyl, il est naturel de se demander s'il existe un résultat similaire pour les modules d'algèbres de Hecke.

Nous introduisons maintenant la configuration pour les groupes de Weyl affines et les algèbres de Hecke affines. Désormais, nous supposons que R est irréductible pour simplifier. Définissons le groupe de Weyl affine $W_{\mathrm{aff}} = W \ltimes \mathbb{Z}R$ (*resp.* le groupe de Weyl affine étendu $W(\mathcal{R}) = W \ltimes X$) comme le produit semi-direct du groupe de Weyl fini W par le réseau $\mathbb{Z}R$ (*resp.* X). Soit $w = w_{\mathrm{fin}} t_\lambda$ la décomposition de w en tant qu'élément de $W(\mathcal{R})$. Ici, w_{fin} désigne la partie du groupe de Weyl fini et t_λ, $\lambda \in X$, désigne la partie de translation associée à $\lambda \in X$. Nous savons que $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$ est un système de Coxeter avec W_{aff} et l'ensemble des générateurs $S_{\mathrm{aff}} := S \bigcup \{\text{une réflexion affine } s_0\}$. Il existe une décomposition en produit semi-direct de $W(\mathcal{R})$ comme $\Omega \ltimes W_{\mathrm{aff}}$ où Ω est le stabilisateur de "l'alcôve fondamentale" déterminée par S_{aff}. Soit ℓ la fonction de longueur pour le système de Coxeter $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$: elle s'étend naturellement à $W(\mathcal{R})$ en exigeant que $\ell(w) = 0$ si $w \in \Omega$. Ainsi, nous pouvons écrire $w = \gamma\tau$, avec $\gamma \in \Omega$ de longueur 0 et τ un produit de $\ell(w)$ éléments de S_{aff}.

Nous désignons par $\mathcal{H}$ l'algèbre de Hecke affine étendue $\mathcal{H}(W(\mathcal{R}), q_s)$ sur un corps fixé K associée à $W(\mathcal{R})$ (voir Définition 3.3.3 pour plus de détails) et des paramètres inégaux $q_s \in K^\times \backslash \{\text{racines de l'unité}\}$, $s \in S$. Nous laissons également $\mathcal{H}_I$ être la sous-algèbre de $\mathcal{H}$ associée à $W(\mathcal{R})_I := W_I \ltimes X$.

Nous définissons le dual d'un $\mathcal{H}$-module (π, M) par

$$D[M] := \sum_{I\subset S} (-1)^{|I|} [\mathrm{Ind}_I(\mathrm{Res}_I\, M)],$$

où Res_I est le foncteur de restriction habituel vers $\mathcal{H}_I$, tandis que le foncteur d'induction Ind_I est défini par $\mathrm{Ind}_I\, N := \mathcal{H} \otimes_{\mathcal{H}_I} N$ pour un $\mathcal{H}_I$-module N (voir Définition 3.3.7 pour plus de détails) et $[M]$ représente l'image de M dans le groupe de Grothendieck des modules de dimension finie sur K.

Notre premier théorème principal est l'analogie de la formule (1.2.1) pour les $\mathcal{H}$-modules

Theorem 1.2.1. *Définissons une action tordue $*$ sur $\mathcal{H}$ comme suit : pour*

$$w = w_{\text{fin}} t_\lambda = w_\Omega t_\mu s_{i_1} s_{i_2} \cdots s_{i_r} \in W(\mathcal{R})$$

posons

$$T_w^* = (-1)^{\ell(w_{\text{fin}})} q(w) T_{w^{-1}}^{-1}$$

où $q(w) = \prod_{k=1}^{r} q_{s_{i_k}}$. Étant donné un $\mathcal{H}$-module (π, M), soit (π^*, M^*) le $\mathcal{H}$-module tel que $M^* = M$ en tant qu'espace vectoriel sur K, équipé de l'action de $\mathcal{H}$: $\pi^*(h)(m) := \pi(h^*)(m)$, $\forall m \in M$ et $\forall h \in \mathcal{H}$. Alors nous avons l'égalité suivante dans le groupe de Grothendieck :

$$D[M] = [M^*].$$

Maintenant, expliquons les étapes principales de la preuve du Théorème 1.2.1. Nous introduisons une somme

$$\tau := \sum_{v \in W} (-1)^{\ell(v)} T_v \otimes T_v^{-1}.$$

En utilisant le complexe de chaîne défini par Deligne-Lusztig dans [27], prouver le Théorème 1.2.1 équivaut à montrer que pour tout $w \in W(\mathcal{R})$,

$$(T_w \otimes 1)\tau = \tau(1 \otimes T_w^*) = \tau(1 \otimes (-1)^{\ell(w_{\text{fin}})} q(w) T_{w^{-1}}^{-1}),$$

où $T_w \otimes 1 \in \mathcal{H} \otimes \mathcal{H}$. Nous prouvons ceci en cinq étapes :

(1) Nous utilisons la présentation d'Iwahori-Matsumoto de T_w pour écrire

$$T_w \otimes 1 = (T_\gamma \otimes 1)(T_{s_{i_1}} \otimes 1)(T_{s_{i_2}} \otimes 1) \cdots (T_{s_{i_k}} \otimes 1),$$

où $w = \gamma s_{i_1} s_{i_2} \cdots s_{i_k}$ avec $\gamma = w_\Omega t_\mu \in \Omega$ et $s_{i_t} \in S_{\text{aff}}$ pour tout $1 \leqslant t \leqslant k$.

(2) Lorsque $s_{i_t} \neq s_0$, nous regroupons les termes dans τ et écrivons ce dernier comme une somme alternée de tels paires :

$$\tau = \sum \pm (T_v \otimes T_v^{-1} - T_{s_{i_t} v} \otimes T_{s_{i_t} v}^{-1}),$$

où le signe $\pm$ dépend de $\ell(v)$. Ceci est fait dans (4.2.4) de 4.2.1.

(3) Lorsque $s_{i_t} = s_0$, les choses sont plus complexes. Nous regroupons les termes dans τ d'une manière différente pour obtenir une somme alternée des paires :

$$\tau = \sum \pm (T_v \otimes T_v^{-1} - T_{s_{\alpha_0} v} \otimes T_{s_{\alpha_0} v}^{-1}),$$

où le signe $\pm$ dépend de $\ell(v)$. Ensuite, nous calculons $T_{s_0}T_v$ en utilisant le lemme du changement de base démontré par S-I. Kato dans [38, Lemma 1.9] et prouvons qu'il est égal à $\bar{T}_{s_0v}$ (voir 3.3.7 pour la définition de $\bar{T}$), sous l'hypothèse que $s_0 \notin \mathcal{L}(A^-v)$ (voir (4.2.1) et (4.2.3) pour la définition). Avec $\ell(s_0) + \ell(v) = \ell(s_0v)$, nous avons $T_{s_0}T_v = T_{s_0v}$, donc $T_{s_0v} = \bar{T}_{s_0v}$ et $T_{s_0v} \otimes T_{s_0v}^{-1} = \bar{T}_{s_0v} \otimes \bar{T}_{s_0v}^{-1} = T_{s_{\alpha_0}v} \otimes T_{s_{\alpha_0}v}^{-1}$.

(4) Nous vérifions que pour tout $\gamma \in \Omega$, T_γ s'entrelace avec τ, d'après [39, Lemma 2].

(5) Nous concluons en mettant les produits de T_{s_i} et des constantes q dans la forme désirée, grâce à la Proposition 4.0.1.

Partie II : La version relative de l'involution par comparaison : le cas fini et le cas affine

La deuxième partie de cette thèse est notre tentative de généraliser la formule d'involution du Théorème 1.2.1 aux algèbres de Hecke attachées à un bloc de Bernstein arbitraire, tout en exigeant que cette généralisation soit compatible avec la dualité d'Aubert-Zelevinsky restreinte à un bloc de Bernstein arbitraire. Plutôt que d'attaquer ce problème directement, nous revenons au cas des groupes finis et commençons par généraliser les résultats de Howlett-Lehrer sur les représentations des algèbres d'endomorphismes finies (algèbres de Hecke). Ce résultat peut être vu comme l'équivalent de la dualité d'Alvis-Curtis-Kawanaka restreinte à une série de Harish-Chandra. Ensuite, nous passons à l'étude du cas p-adique guidés par ce que nous voyons du cas fini, et nous obtenons (6.6.3) comme notre résultat de comparaison avec la dualité d'Aubert-Zelevinsky.

Nous considérons maintenant les groupes de Weyl finis et leurs algèbres de Hecke associées. Soit G un groupe réductif fini $\mathbb{F}_q$-split. Nous fixons un sous-groupe de Borel B, et nous notons son tore maximal par T. À partir de la théorie des BN-paires, nous savons que $(W := N_G(T)/T, S)$ (S est déterminé par B) est un système de Coxeter. Pour un sous-ensemble $I_0 \subset S$, W_{I_0} est le sous-groupe du groupe de Weyl fini (W, S) engendré par $s_\alpha \in I_0$. Les sous-groupes paraboliques standard contenant B sont $P_I := BW_IB$ pour $I \subset S$. Soit Λ une représentation cuspidale irréductible de L_{I_0}. Nous définissons le groupe de ramification $W(\Lambda)$ comme $W(\Lambda) := \{w \in S_{I_0} \mid \chi_\Lambda \circ w = \chi_\Lambda\}$, qui est presque un groupe de réflexion. Nous voulons mentionner que dans le cas fini, la classe de G-conjugaison de la paire (L_{I_0}, Λ) joue le même rôle que le "support inertiel $\mathfrak{s}$" dans le cas p-adique : (L_{I_0}, Λ) (*resp.* $\mathfrak{s}$) détermine un sous-ensemble (*resp.* une sous-catégorie) de toutes les représentations irréductibles (*resp.* $\mathrm{Rep}(G(F))$). R.B. Howlett et G.I. Lehrer [34, Théorème et Corollaire 1] ont prouvé

$$\sum_{I\subset S}(-1)^{|I|}\sum_{w\in W_I\backslash C_{I_0}(I)/W(\Lambda)}\mathrm{Ind}_{W(\Lambda)\cap W_I^w}^{W(\Lambda)}(\mathrm{Res}_{W(\Lambda)\cap W_I^w}^{W(\Lambda)}(\chi)) = \hat{\chi} := (-1)^{|I_0|}(-1)^{\ell_{I_0^\perp}(-)}\chi, \tag{1.2.2}$$

où nous posons $C_{I_0}(I) = \{w \in W \mid wI_0 \subset I\}$ pour tout $I \subset S$ et $\ell_{I_0^\perp}$ est la fonction de longueur de $W(\Lambda)$ associée à un certain système de Coxeter (voir Lemme 5.5.7 et Définition 5.5.8) dans $\langle I_0\rangle^{\perp}$.

Si nous prenons $I_0 = \varnothing$ et remplaçons $W(\Lambda)$ par W, nous voyons que (1.2.2) dégénère en (1.2.1).

Par la comparaison de $\mathrm{Irr}_{\mathbb{C}}(W(\Lambda))$, des séries de Harish-Chandra $\mathrm{Irr}_{\mathbb{C}}(G|(L_{I_0}, \Lambda))$ et des modules simples de l'algèbre de Hecke $E_G(\Lambda) := \mathrm{End}_G(\mathrm{Ind}_{P_{I_0}}^G \Lambda)$ dans la Section 5.6, nous obtenons notre deuxième théorème principale:

Theorem 1.2.2. *Supposons que $W(\Lambda)$ soit véritablement un groupe de réflexions*[1]. *Soit M^* le module M muni de l'action tordue de $E_G(\Lambda)$ définie par*[2].

$$T_w^* = (-1)^{|I_0|}(-q)^{\ell_{I_0^\perp}(w)} T_{w^{-1}}^{-1}.$$

Alors nous avons l'égalité suivante dans le groupe de Grothendieck des modules de $E_G(\Lambda)$

$$\sum_{I\subset S}(-1)^{|I|} \sum_{w\in W_I\backslash C_{I_0}(I)/W(\Lambda)} [\mathrm{Ind}_{E_I'}^{E_G(\Lambda)}[\mathrm{Res}_{E_I'}^{E_G(\Lambda)}(M)]] = [M^*], \tag{1.2.3}$$

où $E_G(\Lambda)$ est défini comme $\mathrm{End}_G(\mathrm{Ind}_{P_{I_0}}^G(\Lambda))$ avec Λ vu comme un module P_{I_0}-module à travers l'élévation naturelle $P_{I_0} \to L_{I_0}$ et E_I' est le sous-algèbre de $E_G(\Lambda)$ engendré par $\{T_v \mid v \in W(\Lambda) \cap W_I^w\}$.

Si nous prenons $I_0 = \varnothing$, alors $C_{I_0}(I) = W$ et $P_{I_0} = P_\varnothing = P_0$ est le sous-groupe parabolique minimal avec $L_{I_0} = T$. Pour Λ égale au caractère trivial de T, nous avons $W(\Lambda) = W$, tandis que $E_G(1) = \mathrm{End}_G(\mathrm{Ind}_{P_0}^G(1))$ est juste l'algèbre de Hecke finie H par rapport à (W,S), et E_I' est le sous-algèbre de H engendré par $\{T_v \mid v \in W_I\} = H_I$. Ainsi, nous voyons que (1.2.3) fournit dans ce cadre une version de (1.2.1) pour les algèbres de Hecke finies.

Motivés par les résultats pour les algèbres de Hecke finies, nous suivons la même approche pour étudier le cas affine. La Section 6.6 est consacrée à la comparaison de la dualité d'Aubert-Zelevinsky du côté du groupe et de l'involution du côté de l'algèbre de Hecke. En suivant [53], nous déduisons deux diagrammes (6.5.4) et (6.5.5) qui donnent les contreparties de l'induction normalisée et des modules de Jacquet sur les catégories de modules à droite sur une certaine algèbre d'endomorphismes. Nous pouvons ainsi étudier l'involution correspondante (voir (6.6.3)) et montrer la relation avec les deux involutions précédentes dans le Théorème 1.2.1 et 1.2.2.

1. Voir l'hypothèse 5.7.1.
2. Dans l'introduction, nous supposons des paramètres égaux pour simplifier notre notation, tandis que le cas général est énoncé et démontré dans le corps de cette thèse.

Partie III : Calculs pour le groupe p-adique G_2

La troisième partie de cette thèse est consacrée à certains calculs de la dualité d'Aubert-Zelevinsky pour les blocs principaux du groupe p-adique deployé G_2 et pour les modules correspondants sur l'algèbre de Hecke affine appropriée, basée sur [8] et [52]. Nous utilisons la description des blocs principaux donnée dans [52] : d'abord, nous introduisons un groupe d'endoscopie $J^{\mathfrak{s}}$ associé au bloc principal $\mathfrak{s}$ tel que la catégorie des modules sur l'algèbre de Hecke-Iwahori $\mathcal{H}(J_{\mathfrak{s}}, 1_J)$ est équivalente aux blocs principaux $\mathrm{Rep}^{\mathfrak{s}}(G)$. Dans le cas de G_2, nous savons d'après [8] que $J^{\mathfrak{s}}$ peut être parmi : $G_2(\mathbb{C})$, $\mathrm{SO}_4(\mathbb{C})$, $\mathrm{SL}_3(\mathbb{C})$ ou $\mathrm{GL}_2(\mathbb{C})$. L'algèbre de Hecke correspondante est alors toujours de rang deux et ses modules sont classifiés par des triples d'indices, voir par exemple [49] et [50]. Suivant la structure de [9, Section 9], nous discutons tous les cas possibles pour les blocs principaux. Ensuite, à travers une étude cas par cas et une adaptation des techniques développées par Muic dans [47], nous calculons les images de tous les facteurs de composition sous l'opérateur de dualité. En comparaison avec [9] et [50], nous obtenons les résultats correspondants pour les modules sur les algèbres de Hecke.

1.3 Quelques projets de recherche

Dans un travail récent [48], K. Ohara a prouvé que pour un type de profondeur zéro (K, ρ) satisfaisant certaines hypothèses, il existe un isomorphisme d'algèbres

$$\mathrm{End}_{G(F)}\left(\text{c-ind}_K^{G(F)}(\rho)\right) \xrightarrow{\sim} \mathrm{End}_{G(F)}\left(I_P^G\left(\text{c-ind}_{L^1}^{L(F)}(\sigma_1)\right)\right). \tag{1.3.1}$$

Il a également établi une identification des algèbres de Hecke affines correspondantes en termes de données racines (voir [48, Théorème 7.15]).

Dans des travaux très récent [1], [2] et [3], J. Adler, J. Fintzen, M. Mishra et K. Ohara ont prouvé le résultat suivant : soit (K, ρ) un type de Kim-Yu pour $\mathrm{Rep}^{\mathfrak{s}}(G(F))$, il existe un sous-groupe de Levi tordu G^0 de G et un sous-groupe de Levi tordu L^0 de G^0, une représentation supercuspidale de profondeur zéro σ_0 de $L^0(F)$ et un type (K_0, ρ_0) pour $\mathrm{Rep}^{\mathfrak{s}_0}\left(G^0\right)$ $\left(\mathfrak{s}_0 := \left[L^0, \sigma_0\right]\right)$ tel que

$$\mathrm{Rep}^{\mathfrak{s}}(G(F)) \cong \mathrm{End}_{G(F)}\left(\text{c-ind}_K^{G(F)}(\rho)\right) \cong \mathrm{End}_{G^0(F)}\left(\text{c-ind}_{K^0}^{G^0(F)}(\rho_0)\right) \cong \mathrm{Rep}^{\mathfrak{s}_0}\left(G^0(F)\right).$$

Un objectif principal de notre projet de recherche est d'obtenir une formule d'involution explicite pour les algèbres d'endomorphismes mentionnées ci-dessus, en termes des relations de générateurs de la base d'Iwahori-Matusumoto. En particulier, nous voulons voir ce qui se passe dans le cas où le groupe R et le 2-cocycle sont tous deux non triviaux.

Notre stratégie générale est la suivante. Notre première étape est de retirer l'Hypothèse 5.7.1 dans le Théorème 5.7.2 pour obtenir le Théorème dans tous les cas généraux ; nous essaierons ensuite de comprendre le travail de [48], où une comparaison détaillée est faite entre les parties de l'algèbre de Hecke affine des algèbres d'endomorphismes de Solleveld et de Morris. Nous clarifierons également les relations de leurs groupes R et de leurs 2-cocycles. Après cela, le travail de J. Adler, J. Fintzen, M. Mishra et K. Ohara nous indique qu'il est suffisant de considérer le cas de profondeur zéro. Ensuite, (2.3.1) nous permet d'apprendre des résultats de Morris sur $\mathrm{End}_{G(F)}\left(I_P^G\left(\text{c-ind}_{L^\circ}^{L(F)}(\sigma_1)\right)\right)$. Comme mentionné dans l'introduction de [45], son algèbre d'endomorphismes "sur la philosophie est implicitement adopté en adaptant la théorie de Howlett-Lehrer" (mais techniquement plus compliqué). Dans notre thèse, nous avons fourni une formule d'involution explicite sous l'Hypothèse 5.7.1 pour les algèbres de Hecke apparaissant dans le travail de Howlett-Lehrer. Par conséquent, nous nous attendons à obtenir une formule explicite dans le cas étudié par Morris, puis pour chaque algèbre d'endomorphismes associée à un bloc arbitraire via le travail de J. Adler, J. Fintzen, M. Mishra et K. Ohara. En chemin, nous prévoyons également de fournir des calculs de certains exemples explicites comme ce que nous avons fait pour G_2.

Dans une direction différente, inspirée par le travail effectué dans l'étude de l'involution dans le cas de G_2, il serait intéressant de voir l'application de cette involution dans la correspondance de Langlands locale. Il semble y avoir de nombreuses applications pertinentes de cette involution dans ce cadre, telles que l'étude des A-paquets, voir [22], [23] et [24], ses questions de fonctorialité associées, voir [32] et l'étude de la correspondance de Langlands locale explicite d'après les méthodes de [9] et [10].

Chapter 2

Introduction in English

2.1 An overview of history

The study of duality for representations with complex coefficients of finite Weyl groups and finite groups of Lie type can be traced back from the 1960s, with contributions from the works of D. Alvis [4], L. Solomon [58], and extending to C.W. Curtis [25], R.B. Howlett & G.I. Lehrer [34] and P. Deligne & G. Lusztig [27] in 1980s. Several years later, influenced by the proof of [27], S-I. Kato [39] defined a duality operator for representations of Hecke algebras associated with a finite or an affine Weyl group with equal parameters, and he expressed this operator in terms of a certain involution on the Hecke algebra. Inspired by the Hecke algebra case, A-M. Aubert defined in 1995 [7] a duality (now known as Aubert-Zelevinsky duality) on the Grothendieck group of the category of smooth, finite length, complex representations of a p-adic group, and she proved several properties of this duality.

To be more precise, let us introduce more notations. Let F be a non-Archimedean local field and G be a connected reductive group over F. The category $\mathrm{Rep}(G(F))$ of smooth complex representations of $G(F)$ has a Bernstein decomposition as a product of subcategories (called *Bernstein blocks*)

$$\prod_{\mathfrak{s}\in\mathfrak{B}(G(F))} \mathrm{Rep}^{\mathfrak{s}}(G(F)),$$

parametrized by inertial supports $\mathfrak{s} = [L, \sigma]_G$ where L is a Levi subgroup of G and σ is an irreducible supercuspidal representation of $L(F)$.

Let us first recall the approach taken by J. Bernstein [13], A. Roche [52], M. Solleveld [57] (among others) to study $\mathrm{Rep}^{\mathfrak{s}}(G(F))$. We choose a parabolic subgroup P such that $P = LU$. Let σ_1 be an irreducible subrepresentation of $\sigma|_{L^1}$ where L^1 denotes the intersection of the kernels of unramified characters of $L(F)$. Then

$$\mathrm{Rep}^{\mathfrak{s}}(G(F)) \cong \mathrm{Mod}\text{-}\mathrm{End}_{G(F)}\left(I_P^G\left(\text{c-ind}_{L^1}^{L(F)}(\sigma_1)\right)\right),$$

where I_P^G denotes the normalized parabolic induction from $P(F)$ to $G(F)$ and $\text{c-ind}_{L^1}^{L(F)}(\sigma_1)$ is the small progenerator. We know from M. Solleveld's work [57, Theorem 10.9] that the endomorphism algebra $\text{End}_{G(F)}\left(I_P^G\left(\text{c-ind}_{L^1}^{L(F)}(\sigma_1)\right)\right)$ is an extension of an affine Hecke algebra by a group algebra twisted by a 2-cocycle.

Another approach to study $\text{Rep}^{\mathfrak{s}}(G(F))$ is the theory of types, developed by C.J. Bushnell & P.C. Kutzko [19], A. Roche [31], S. Stevens [59] (among others). Under certain assumptions, we know there is an equivalence of categories

$$\text{Rep}^{\mathfrak{s}}(G(F)) \cong \text{Mod-End}_{G(F)}\left(\text{c-ind}_K^{G(F)}(\rho)\right),$$

if (K,ρ) is a $\mathfrak{s}$-type. For (K,ρ) a depth-zero type, L. Morris proved in [46] that the endomorphism algebra $\text{End}_{G(F)}\left(\text{c-ind}_K^{G(F)}(\rho)\right)$ is isomorphic to an extension of an affine Hecke algebra by a group algebra twisted by a 2-cocycle.

Moreover, S-I. Kato's involution can be seen, under some assumptions, the counterpart of the Aubert-Zelevinsky duality on the category of modules over some Hecke algebra. Our goal is to work out the formula for the involution on more general Hecke algebras, namely those corresponding to an arbitrary block $\text{Rep}^{\mathfrak{s}}(G)$.

2.2 Main results of this thesis

This thesis is divided into three parts. The first part is a generalization of S-I. Kato's results to the unequal parameter case, furthermore along the way we add more details to the proofs and correct the errors. The second part starts with the involution theorem for finite Hecke algebras that appear in Howlett-Lehrer's theory, which serves as a stepping stone for the study of involution for an arbitrary Bernstein block. Inspired by the finite Hecke algebra case, we obtain the left hand side of involution for an arbitrary Bernstein block. In third part, we studied the Aubert-Zelevinsky duality on the principal series of the exceptional group G_2 and the corresponding involution for Hecke algebras based on the recent works of Aubert-Xu in [9] and [10].

Part I: Involution for unequal parameter generalized affine Hecke algebras

We now present a more detailed introduction for the first part. Let $\mathcal{R} := (X, R, Y, R^\vee)$ be a root datum satisfying axioms listed in [56, Section 1.3] and Δ denote a set of simple roots of R. We define s_α as the reflection with respect to the hyperplane orthogonal to α. Let W be a finite Weyl group generated by the set of simple reflections $S = \{s_\alpha : \alpha \in \Delta\}$. For any element $w \in W$, we introduce its length $\ell(w)$ as the smallest number such that w can be written as a product of $\ell(w)$ simple reflections, and $\text{sgn}(w) := (-1)^{\ell(w)}$. Let w_0 denote the longest element of W. For $I \subset S$ a subset of S, let $|I|$ denote the cardinality of I. Let W_I denote the subgroup of W generated by I. Also let $\text{Res}_{W_I}^W$ (*resp.* $\text{Ind}_{W_I}^W$)

denote the restriction (*resp.* induction functor) for representations of the finite groups W and W_I. In [58, Theorem 2], L. Solomon proves that for any character χ of W, one has:

$$\sum_{I \subset S} (-1)^{|I|} \operatorname{Ind}_{W_I}^{W} \operatorname{Res}_{W_I}^{W}(\chi) = \widehat{\chi} := \chi \otimes \operatorname{sgn}. \tag{2.2.1}$$

From the perspective that the Hecke algebras can be regarded as a "deformed" group algebras associated with Weyl groups, it is natural to wonder whether there exists a similar result for Hecke algebras modules.

We now introduce the setup for affine Weyl groups and affine Hecke algebras. From now on, we assume that R is irreducible for simplicity. Let us define the affine Weyl group $W_{\text{aff}} = W \ltimes \mathbb{Z}R$ (*resp.* extended affine Weyl group $W(\mathcal{R}) = W \ltimes X$) as the semi-direct product of finite Weyl group W by the lattice $\mathbb{Z}R$ (*resp.* X). Let $w = w_{\text{fin}} t_\lambda$ be the decomposition for w as an element in $W(\mathcal{R})$. Here w_{fin} denotes the finite Weyl group part and t_λ, $\lambda \in X$ denotes the translation part associated with some $\lambda \in X$. We know that $(W_{\text{aff}}, S_{\text{aff}})$ is a Coxeter system with W_{aff} and the set of generators $S_{\text{aff}} := S \bigcup \{\text{an affine reflection } s_0\}$. There exists a semi-direct product decomposition of $W(\mathcal{R})$ as $\Omega \ltimes W_{\text{aff}}$ where Ω is the stabilizer of "fundamental alcove" determined by S_{aff}. Let ℓ be the length function for the Coxeter system $(W_{\text{aff}}, S_{\text{aff}})$: it extends naturally to $W(\mathcal{R})$ by requiring $\ell(w) = 0$ if $w \in \Omega$. We may thus write $w = \gamma\tau$, with $\gamma \in \Omega$ of length 0 and τ a product of $\ell(w)$ elements of S_{aff}.

We denote by $\mathcal{H}$ as the extended affine Hecke algebra $\mathcal{H}(W(\mathcal{R}), q_s)$ over a fixed field K associated with $W(\mathcal{R})$ (see Definition 3.3.3 for more details) and unequal parameters $q_s \in K^\times \backslash \{\text{roots of unity}\}$, $s \in S$. We also let $\mathcal{H}_I$ be the subalgebra of $\mathcal{H}$ associated with $W(\mathcal{R})_I := W_I \ltimes X$.

We define the dual of a $\mathcal{H}$-module (π, M) by

$$D[M] := \sum_{I \subset S} (-1)^{|I|} [\operatorname{Ind}_I(\operatorname{Res}_I M)],$$

where Res_I is the usual restriction functor to $\mathcal{H}_I$, while the induction functor Ind_I is defined by $\operatorname{Ind}_I N := \mathcal{H} \otimes_{\mathcal{H}_I} N$ for an $\mathcal{H}_I$-module N (see Definition 3.3.7 for more details) and $[M]$ denotes the image of M in the Grothendieck group of finite dimensional modules over K.

Our first principal theorem is the analogue of the formula (2.2.1) for the $\mathcal{H}$-modules:

Theorem 2.2.1. *Define a twisted action $*$ on $\mathcal{H}$ as follows: for*

$$w = w_{\text{fin}} t_\lambda = w_\Omega t_\mu s_{i_1} s_{i_2} \cdots s_{i_r} \in W(\mathcal{R})$$

set

$$T_w^* = (-1)^{l(w_{\text{fin}})} q(w) T_{w^{-1}}^{-1}$$

where $q(w) = \prod_{k=1}^{r} q_{s_{i_k}}$. *Given* $\mathcal{H}$*-module* (π, M)*, let* (π^*, M^*) *be the* $\mathcal{H}$*-module such that* $M^* = M$ *as* K*-vector space, equipped with the* $\mathcal{H}$*-action* $\pi^*(h)(m) := \pi(h^*)(m)$*,* $\forall m \in M$ *and* $\forall h \in \mathcal{H}$*. Then we have the following equality in the Grothendieck group:*

$$D[M] = [M^*].$$

Now we explain the main steps proof for Theorem 2.2.1. We introduce a sum

$$\tau := \sum_{v \in W} (-1)^{\ell(v)} T_v \otimes T_v^{-1}.$$

Using the chain complex defined by Deligne-Lusztig in [27], proving Theorem 2.2.1 is equivalent to prove that for any $w \in W(\mathcal{R})$,

$$(T_w \otimes 1)\tau = \tau(1 \otimes T_w^*) = \tau(1 \otimes (-1)^{l(w_{\mathrm{fin}})} q(w) T_{w^{-1}}^{-1}),$$

where $T_w \otimes 1 \in \mathcal{H} \otimes \mathcal{H}$. We prove this in five steps:

(1) We use the Iwahori-Matsumoto presentation of T_w to write

$$T_w \otimes 1 = (T_\gamma \otimes 1)(T_{s_{i_1}} \otimes 1)(T_{s_{i_2}} \otimes 1) \cdots (T_{s_{i_k}} \otimes 1),$$

where $w = \gamma s_{i_1} s_{i_2} \cdots s_{i_k}$ with $\gamma = w_\Omega t_\mu \in \Omega$ and $s_{i_t} \in S_{\mathrm{aff}}$ for any $1 \leqslant t \leqslant k$.

(2) When $s_{i_t} \neq s_0$, we pair up terms in τ and write the latter as alternating sum of such pairs:

$$\tau = \sum \pm (T_v \otimes T_v^{-1} - T_{s_{i_t} v} \otimes T_{s_{i_t} v}^{-1}),$$

where the sign $\pm$ depends on $\ell(v)$. This is done in (4.2.4) of 4.2.1.

(3) When $s_{i_t} = s_0$, things are harder. We pair up terms in τ in a different way to obtain an alternating sum of the pairs:

$$\tau = \sum \pm (T_v \otimes T_v^{-1} - T_{s_{\alpha_0} v} \otimes T_{s_{\alpha_0} v}^{-1}),$$

where the sign $\pm$ depends on $\ell(v)$. Then we compute $T_{s_0} T_v$ using the base change lemma proved by S-I. Kato in [38, Lemma 1.9] and prove that it equals $\bar{T}_{s_0 v}$ (see 3.3.7 for definition of $\bar{T}$), under the assumption $s_0 \notin \mathcal{L}(A^- v)$ (see (4.2.1) and (4.2.3) for definition). By $\ell(s_0) + \ell(v) = \ell(s_0 v)$, we have $T_{s_0} T_v = T_{s_0 v}$, so $T_{s_0 v} = \bar{T}_{s_0 v}$ and $T_{s_0 v} \otimes T_{s_0 v}^{-1} = \bar{T}_{s_0 v} \otimes \bar{T}_{s_0 v}^{-1} = T_{s_{\alpha_0} v} \otimes T_{s_{\alpha_0} v}^{-1}$.

(4) We check that for any $\gamma \in \Omega$, T_γ intertwines with τ, following [39, Lemma 2].

(5) We conclude by arranging products of T_{s_i} and q-constants in the desired form, thanks to Proposition 4.0.1.

Part II: The relative version of involution via comparison: the finite case and the affine case

The second part of this thesis is our attempt to generalize the involution formula of Theorem 2.2.1 to the Hecke algebras attached to an arbitrary Bernstein block, and we require such generalization compatible with Aubert-Zelevinsky duality restricted to an arbitrary Bernstein block. Instead of attacking this problem directly, we step back to the finite group case and start by generalizing Howlett-Lehrer's results for representations of the finite endomorphism algebras (Hecke algebra). This result can be viewed as the counterpart of Alvis-Curtis-Kawanaka duality restricted to a Harish-Chandra series. Then we proceed to study the p-adic case guided by what we see from finite case, and we obtain (6.6.3) as our result of comparison with Aubert-Zelevinsky duality.

We now consider the finite Weyl groups and their associated Hecke algebras. Let G be a $\mathbb{F}_q$-split finite reductive group. We fix a Borel subgroup B, denote its maximal torus by T. From the theory of BN-pairs we know $(W := N_G(T)/T, S)$ (S is determined by B) is a Coxeter system. For a subset $I_0 \subset S$, let W_{I_0} be the subgroup of a finite Weyl group (W, S) generated by $s_\alpha \in I_0$, the standard parabolic subgroups containing B are $P_I := BW_IB$ for $I \subset S$. Let Λ denote an irreducible cuspidal representation of L_{I_0}, χ_Λ denote its character. We define the ramification group $W(\Lambda)$ as $W(\Lambda) := \{w \in S_{I_0} \mid \chi_\Lambda \circ w = \chi_\Lambda\}$ which is almost a reflection group. We want to mention that in the finite case, the G-conjugacy class of the pair (L_{I_0}, Λ) plays the same role as "inertial support $\mathfrak{s}$" in p-adic case: (L_{I_0}, Λ) (*resp.* $\mathfrak{s}$) determines a subset (*resp.* subcategory) of all irreducible representations (*resp.* $\mathrm{Rep}(G(F))$). R.B. Howlett and G.I. Lehrer [34, Theorem and Corollary 1] proved

$$\sum_{I \subset S} (-1)^{|I|} \sum_{w \in W_I \backslash C_{I_0}(I)/W(\Lambda)} \mathrm{Ind}_{W(\Lambda) \cap W_I^w}^{W(\Lambda)} (\mathrm{Res}_{W(\Lambda) \cap W_I^w}^{W(\Lambda)}(\chi)) = \hat{\chi} := (-1)^{|I_0|}(-1)^{\ell_{I_0^\perp}(-)}\chi, \tag{2.2.2}$$

where we set $C_{I_0}(I) = \{w \in W \mid wI_0 \subset I\}$ for any $I \subset S$ and $\ell_{I_0^\perp}$ is the length function of $W(\Lambda)$ associated with some Coxeter system (see Lemma 5.5.7 and Definition 5.5.8) in $< I_0 >^\perp$.

If we take $I_0 = \varnothing$ and replace $W(\Lambda)$ by W, we see that (2.2.2) degenerates to (2.2.1).

By the comparison of $\mathrm{Irr}_{\mathbb{C}}(W(\Lambda))$, Harish-Chandra series $\mathrm{Irr}_{\mathbb{C}}(G|(L_{I_0}, \Lambda))$ and the simple modules of the Hecke algebra $E_G(\Lambda) := \mathrm{End}_G(\mathrm{Ind}_{P_{I_0}}^G \Lambda)$ in Section 5.6, we have our second main theorem:

Theorem 2.2.2. *Assume that $W(\Lambda)$ is truly a reflection group*[1]*. Let M^* denote the*

1. See Assumption 5.7.1.

module M endowed with the twisted action of $E_G(\Lambda)$ defined by[2]

$$T_w^* = (-1)^{|I_0|}(-q)^{\ell_{I_0^\perp}(w)} T_{w^{-1}}^{-1}.$$

Then we have the following equality in the Grothendieck group of $E_G(\Lambda)$-modules:

$$\sum_{I \subset S} (-1)^{|I|} \sum_{w \in W_I \backslash C_{I_0}(I)/W(\Lambda)} [\mathrm{Ind}_{E_I'}^{E_G(\Lambda)} [\mathrm{Res}_{E_I'}^{E_G(\Lambda)}(M)]] = [M^*], \tag{2.2.3}$$

where $E_G(\Lambda)$ is defined as $\mathrm{End}_G(\mathrm{Ind}_{P_{I_0}}^G(\Lambda))$ with Λ viewed as a P_{I_0}-module through the natural lifting $P_{I_0} \to L_{I_0}$ and E_I' is the subalgebra of $E_G(\Lambda)$ spanned by $\{T_v \mid v \in W(\Lambda) \cap W_I^w\}$.

If we take $I_0 = \varnothing$, then $C_{I_0}(I) = W$ and $P_{I_0} = P_\varnothing = P_0$ is the minimal parabolic subgroup with $L_{I_0} = T$. For Λ equals the trivial character of T, we have $W(\Lambda) = W$, while $E_G(1) = \mathrm{End}_G(\mathrm{Ind}_{P_0}^G(1))$ is just the finite Hecke algebra H with respect to (W, S), and E_I' is the subalgebra of H spanned by $\{T_v \mid v \in W_I\} = H_I$. Hence, we see that (2.2.3) provides in this setting a version of (2.2.1) for finite Hecke algebra.

Motivated by the results for finite Hecke algebras, we follow the same approach to study the affine case. The Section 6.6 is devoted to comparison of the Aubert-Zelevinsky duality on group side and the involution on the Hecke algebra side. Following [53], we deduce two diagrams (6.5.4) and (6.5.5) that give the courterparts of the normalized induction and of Jacquet modules on the categories of right modules over some endomorphism algebra. We can thus study the corresponding involution (see (6.6.3)) and show the relation with the two previous involutions in Theorem 2.2.1 and 2.2.2.

Part III: Computations for the p-adic group G_2

The third part of this thesis is devoted to some computations of the Aubert-Zelevinsky duality for the principal blocks of split p-adic group G_2 and for the corresponding modules over the suitable affine Hecke algebra based on [8] and [52]. We use the description of principal blocks given in [52]: first, we introduce an endoscopy group $J^{\mathfrak{s}}$ that is associated to the principal block $\mathfrak{s}$ such that the category of modules over the Iwahori-Hecke algebra $\mathcal{H}(J_{\mathfrak{s}}, 1_J)$ is equivalent to the principal blocks $\mathrm{Rep}^{\mathfrak{s}}(G)$. In the G_2 case, we know from [8] that $J^{\mathfrak{s}}$ are among: $G_2(\mathbb{C})$, or $\mathrm{SO}_4(\mathbb{C})$, or $\mathrm{SL}_3(\mathbb{C})$ or $\mathrm{GL}_2(\mathbb{C})$. The corresponding Hecke algebra is then always of rank two and its modules are classified by indexing triples, see for instance [49] and [50]. Following the structure of [9, Section 9], we discuss all possible cases for the principal blocks. Then through a case-by-case study and an adaptation of the techniques developed by Muic in [47], we compute the images of all composition factors

2. In the introduction, we assume equal parameters to simplify our notation, while the general case is stated and proved in the body of this thesis.

under the duality operator. By comparison with [9] and [50], we get the corresponding results for modules over the Hecke algebras.

2.3 Some research projects

In a recent work [48], K. Ohara proved that for a depth-zero type (K, ρ) satisfying certain assumptions, there is an isomorphism of algebras

$$\mathrm{End}_{G(F)}\left(\text{c-ind}_K^{G(F)}(\rho)\right) \xrightarrow{\sim} \mathrm{End}_{G(F)}\left(I_P^G\left(\text{c-ind}_{L^1}^{L(F)}(\sigma_1)\right)\right). \tag{2.3.1}$$

He also proved an identification of the corresponding affine Hecke algebras in terms of root data (see [48, Theorem 7.15]).

In the very recent series of articles [1], [2] and [3], J. Adler, J. Fintzen, M. Mishra and K. Ohara proved the following result: let (K, ρ) be a Kim-Yu type for $\mathrm{Rep}^{\mathfrak{s}}(G(F))$, there exists a twisted Levi subgroup G^0 of G and a twisted Levi subgroup L^0 of G^0, a depth-zero supercuspidal representation σ_0 of $L^0(F)$ and a type (K_0, ρ_0) for $\mathrm{Rep}^{\mathfrak{s}_0}\left(G^0\right)$ $\left(\mathfrak{s}_0 := \left[L^0, \sigma_0\right]\right)$ such that

$$\mathrm{Rep}^{\mathfrak{s}}(G(F)) \cong \mathrm{End}_{G(F)}\left(\text{c-ind}_K^{G(F)}(\rho)\right) \cong \mathrm{End}_{G^0(F)}\left(\text{c-ind}_{K^0}^{G^0(F)}(\rho_0)\right) \cong \mathrm{Rep}^{\mathfrak{s}_0}\left(G^0(F)\right).$$

One main goal of our research project is to obtain an explicit involution formula for the above mentioned endomorphism algebras, in terms of the generator relations of the Iwahori-Matusumoto basis. Particularly, we want to see what happens to the case where the R-group and the 2-cocycle are both nontrivial.

Our general strategy is as follows. Our first step is to remove Assumption 5.7.1 in Theorem 5.7.2 to get the Theorem for all general cases; then we will try understand the work of [48], where a detailed comparison is made between affine Hecke algebra parts of the endomorphism algebra of Solleveld, and the one of Morris. We will also make clear the relations of their R-groups and 2-cocycles. After that, the work of J. Adler, J.Fintzen, M. Mishra and K. Ohara tells us it is sufficient to consider the depth-zero case. Then (2.3.1) enable us to learn from Morris results about $\mathrm{End}_{G(F)}\left(I_P^G\left(\text{c-ind}_{L^\circ}^{L(F)}(\sigma_1)\right)\right)$. As mentioned in the introduction of [45], his endomorphisms algebra "on the philosophy is implicitly embraced by adapting Howlett-Lehrer theory" (but technically more complicated). In our thesis, we have provided an explicit involution formula under the Assumption 5.7.1 for the Hecke algebras appearing in Howlett-Lehrer's work. Hence, we expect to obtain an explicit formula in the case studied by Morris, and then for any endomorphism algebra associated with an arbitrary block via the work of J. Adler, J. Fintzen, M. Mishra and K. Ohara. Along the way, we also expect to provide computations of certain explicit examples like what we have done for G_2.

In a different direction, inspired by the work done in studying the involution in the G_2

case, it would be interesting to see the application of this involution in the local Langlands correspondence. There seem to be many relevant applications of this involution within this framework, such the study of A-packets, see [22], [23] and [24], its related functoriality questions, see [32] and the study of explicit local Langlands correspondence following the methods in [9] and [10].

Chapter 3

General setup

3.1 Notation and conventions

$\mathrm{Card}(S)$: the number of elements in a set S.
F: a non-archimedean local field (starting in Chapter 6).
$\mathfrak{o}$: ring of integer of F.
ϖ_F: a fixed prime element of F.
ν_F: normalized absolute value of F.
$\mathfrak{p}$: prime ideal in $\mathfrak{o}$, residue field $\mathbb{F}_q = \mathfrak{o}/\mathfrak{p}$, $q = \mathrm{Card}(\mathfrak{o}/\mathfrak{p})$.
Let R be a ring with identity, R-Mod (Mod-R) denote the category of left (*resp.* right) unital R-modules.

3.2 Root systems and affine Weyl groups

A quadruple $\mathcal{R} := (X, R, Y, R^\vee)$ is called *root datum* if the following conditions are met:

— X and Y are lattices of finite rank, with a perfect pairing $\langle \cdot, \cdot \rangle \to \mathbb{Z}$.
— R is the root system in X.
— $R^\vee$ is the coroot (dual root) system such that $\langle \alpha, \alpha^\vee \rangle = 2$.
— For every $\alpha \in R$, $x \in X$, $s_\alpha(x) := x - \langle x, \alpha^\vee \rangle \alpha$ acts on affine space $E := X \otimes \mathbb{R}$ and stabilizes R.
— For every $\alpha^\vee \in R^\vee$, $u \in Y$, $s_\alpha^\vee(u) := u - \langle \alpha, u \rangle \alpha^\vee$ acts on affine space $E^\vee := Y \otimes \mathbb{R}$ and stabilizes $R^\vee$.

For example, if we start with a complex reductive group G with maximal torus T, and take $X = \mathrm{Hom}(T, \mathbb{C}^\times)$, $Y = \mathrm{Hom}(\mathbb{C}^\times, T)$, R the root system of (G, T), $R^\vee$ the coroot system then we get a root datum.

Let Δ be a set of simple roots, and R^+ the positive roots with respect to Δ.

Let $W = W(R)$ denote the group generated by s_α for $\alpha \in R$, and $S = \{s_\alpha : \alpha \in \Delta\}$,

(W,S) is a finite Coxeter system. Let us denote the longest element of W by w_0. W can be identified with a subgroup of $\mathrm{GL}(X\otimes\mathbb{R})$ generated by $\{s_\alpha \mid \alpha \in S\}$ or a subgroup of $\mathrm{GL}(Y\otimes\mathbb{R})$ generated by $\{s_\alpha^\vee \mid \alpha\in S\}$, and the pairing $\langle\cdot,\cdot\rangle$ extends naturally to $X\otimes\mathbb{R}$ and $Y\otimes\mathbb{R}$. Let $W(\mathcal{R}) := W \ltimes X$ be the extended affine Weyl group, it contains a normal subgroup W_{aff}, which is defined as $W\ltimes \mathbb{Z}R$. We will mainly consider the (extended) affine Weyl group acting on $X\otimes\mathbb{R}$. Let $R^\vee_{\max}$ be the set of maximal elements of $R^\vee$, with respect to the dual base $\Delta^\vee$. For simplicity, in the proofs we will assume R is irreducible. Under this assumption, let $\alpha_0^\vee$ be the unique maximal coroot and $s_0 = s_{\alpha_0}t_{-\alpha_0}$, where s_α $(\alpha\in R)$ is the reflection with respect to α and t_x $(x\in X)$ is translation by x. Define $S_{\text{aff}} := S\cup\{s_0\}$, we know that $(W_{\text{aff}}, S_{\text{aff}})$ is a Coxeter system.

The $W(\mathcal{R})$ can be seen as acting on the affine space $X\otimes\mathbb{R}$ by translation and reflections extending the action on X. The hyperplanes $H_{\alpha,n} := \{x\in X\otimes\mathbb{R} \mid \langle x,\alpha^\vee\rangle = n\}$ are $W(\mathcal{R})$ stable and divides $X\otimes\mathbb{R}$ into alcoves. The choice of Δ determines a fundamental alcove A_0 in $X\otimes\mathbb{R}$, the unique alcove contained in the positive Weyl chamber (with respect to Δ), such that $0\in\overline{A}_0$. Put Ω as the stablizer of A_0 in $W(\mathcal{R})$. We have the splitting $W(\mathcal{R}) = \Omega\ltimes W_{\text{aff}}$. We have a formula of the length function for $w := w_{\text{fin}}t_x \in W(\mathcal{R})$:

The length function for $W(\mathcal{R})$, $\ell_{W(\mathcal{R})} : W(\mathcal{R}) = W\ltimes X \to \mathbb{Z}_{\geqslant 0}$ is given by

$$\ell_{W(\mathcal{R})}(w_{\text{fin}}t_x) = \sum_{\alpha\in R^+\cap w_{\text{fin}}^{-1}(R^+)} |\langle x,\alpha^\vee\rangle| + \sum_{\alpha\in R^+\cap w_{\text{fin}}^{-1}(-R^+)} |1+\langle x,\alpha^\vee\rangle|. \tag{3.2.1}$$

When no confusion is caused, we usually omit the sub-index of $\ell_{W(\mathcal{R})}$.

Remark. There are different definitions for "affine Weyl group" in the literature: [17] and [43] define it as the subgroup of $\mathrm{Aff}(Y\otimes\mathbb{R})$ generated by all the affine reflections $r_{\alpha,k}(y) = y-(\langle\alpha,y\rangle-k)\alpha^\vee$ with $\alpha\in R$, $k\in\mathbb{Z}$ ([48] used $\mathcal{R}^\vee$ instead of $\mathcal{R}$). [38], [56] and we define it as the subgroup of $\mathrm{Aff}(X\otimes\mathbb{R})$ generated by all the affine reflections $r_{\alpha,k}(x) = x-(\langle x,\alpha^\vee\rangle-k)\alpha$ with $\alpha\in R$, $k\in\mathbb{Z}$. Most of results in these articles/books can be used by making some adaptions, but notice that when we consider non simply-laced examples from [17] their Weyl groups should be of the type determined by $R^\vee$.

3.3 Affine Hecke algebras

Let K be an arbitrary field of characteristic 0, $(\mathbf{W},\mathbf{S})$ be a Coxeter system ((W,S) or $(W_{\text{aff}},S_{\text{aff}})$). Let $\mathbf{q} = (q_s)_{s\in\mathbf{S}}$ be set of indeterminates satisfying $q_s = q_t$ whenever s and t are conjugate.

Definition 3.3.1 (Generic Hecke algebra)**.** The generic Hecke algebra $\mathcal{H}_{\mathbf{q}}(\mathbf{W},\mathbf{S})$ is the $\mathbb{Z}[\mathbf{q},\mathbf{q}^{-1}]$-algebra generated by $\{T_s\}_{s\in\mathbf{S}}$ satisfying

$$(T_s+1)(T_s-q_s) = 0, \text{ for } s\in\mathbf{S} \text{ (quadratic relation)}, \tag{3.3.1}$$

$$T_sT_tT_s\cdots = T_tT_sT_t\cdots \text{for all } sts\ldots = tst\ldots \in \mathbf{W} \text{ (braid relations).} \tag{3.3.2}$$

We now extend the definition of indeterminates and specialize q_s. Let λ, $\lambda^* : R \to K$ be functions such that

(1) if α, $\beta \in R$ are W-conjugate, then $\lambda(\alpha) = \lambda(\beta)$ and $\lambda^*(\alpha) = \lambda^*(\beta)$,

(2) if $\alpha^\vee \notin 2Y$, then $\lambda^*(\alpha) = \lambda(\alpha)$.

For $\alpha \in R$, we require $q_{s_\alpha} = q^{\lambda(\alpha)}$ and if $\alpha^\vee \in R^\vee_{\max}$, we write $q_{s'_\alpha} = q^{\lambda^*(\alpha)}$ where $s'_\alpha = s_\alpha t_{-\alpha}$. For $w = \gamma\tau \in \Omega \ltimes W_{\text{aff}}$, we define $q(w)$ as the function $q(w) = q(\tau) = \prod_{s \in S_{\text{aff}}} q_s^{n_s}$ with n_s equals the multiplicity of s in a reduced decomposition of w with respect to S_{aff}.

Definition 3.3.2 (Finite Hecke algebra)**.** For the finite Coxeter system (W, S) and parameters $q_s \in K^\times \backslash \{\text{roots of unity}\}$, define the finite Hecke algebra $H = H(W, q_s)$ as the K-algebra with basis T_w, $w \in W$ satisfying:

$$(T_s + 1)(T_s - q_s) = 0, \text{ for } s \in S, \tag{3.3.3}$$

$$T_w \cdot T_{w'} = T_{ww'}, \text{ if } \ell(w) + \ell(w') = \ell(ww'). \tag{3.3.4}$$

Definition 3.3.3 (Iwahori-Matsumoto)**.** For $q_s \in K^\times \backslash \{\text{roots of unity}\}$, define the extended affine Hecke algebra $\mathcal{H} = \mathcal{H}(W(\mathcal{R}), q_s)$ as the K-algebra with basis T_w, $w \in W(\mathcal{R})$ satisfying:

$$(T_s + 1)(T_s - q_s) = 0, \text{ for } s \in S_{\text{aff}}, \tag{3.3.5}$$

$$T_w \cdot T_{w'} = T_{ww'}, \text{ if } \ell(w) + \ell(w') = \ell(ww'). \tag{3.3.6}$$

Now we begin to introduce the Bernstein-Lusztig presentations. Let $\{\theta_x \mid x \in X\}$ be the standard basis of $\mathbb{C}[X]$.

Definition 3.3.4 (Bernstein-Lusztig)**.** The algebra $\mathcal{H}(\mathcal{R}, \lambda, \lambda^*, q)$ is the vector space $K[X] \otimes_K \mathcal{H}(W, q)$ with the multiplication rules:

(1) $K[X]$ and $H(W, q)$ are embedded as subalgebras,

(2) for $\alpha \in \Delta$ and $x \in X$:

$$\theta_x T_{s_\alpha} - T_{s_\alpha}\theta_{s_\alpha(x)} = ((q^{\lambda(\alpha)} - 1) + \theta_{-\alpha}(q^{(\lambda(\alpha)+\lambda^*(\alpha))/2} - q^{(\lambda(\alpha)-\lambda^*(\alpha))/2}))\frac{\theta_x - \theta_{s_\alpha(x)}}{\theta_0 - \theta_{-2\alpha}}.$$

Denote the Iwahori-Hecke algebra associated with $(\mathbb{Z}R, R, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}R, \mathbb{Z}), R^\vee, \Delta)$ by $\mathcal{H}(W_{\text{aff}}, q_s)$ with $q_s \in \mathbb{C}$. There is a group action of Ω on $\mathcal{H}(W_{\text{aff}}, q_s)$ given by:

$$T_{\omega w \omega^{-1}} = \omega \cdot T_w \cdot \omega^{-1}$$

The crossed product algebra $\mathcal{H}(W_{\text{aff}}, q_s) \rtimes \Omega$ is the vector space $\mathcal{H}(W_{\text{aff}}, q_s) \otimes K[\Omega]$ equipped with the above group action and the Hecke algebra structure of $\mathcal{H}(W_{\text{aff}}, q_s)$. We have

$$\Omega \ltimes \mathcal{H}(W_{\text{aff}}, q_s) \cong \mathcal{H}(W(\mathcal{R}), q_s).$$

See [43, Section 3] for more details. From [56] and [43, Section 3], we have the following:

Theorem 3.3.5 (Bernstein). *Pick $\lambda(\alpha), \lambda^*(\alpha)$ such that $q_{s_\alpha} = q^{\lambda(\alpha)}$ for all $\alpha \in R$ and $q_{s'_\alpha} \in q^{\lambda^*(\alpha)}$ when $\alpha^\vee \in R^\vee_{\text{max}}$. Then there exists a unique algebra isomorphism $\mathcal{H}(W_{\text{aff}}, q_s) \rtimes \Omega \to \mathcal{H}(\mathcal{R}, \lambda, \lambda^*, q_s)$ such that:*

(1) it is the identity on $\mathcal{H}(W, q)$,

(2) for $x \in \mathbb{Z}R^\vee \cap X_{\text{dom}}$, its sends $q(t_x)^{-1/2} T_{t_x}$ to θ_x.

For any $\mu \in X$ denote for simplicity that $T_\mu = T_{t_\mu}$. Let us define $\bar{T}_x$, $x \in X$ as in Section 2.6 of [43]:

$$\bar{T}_x := T_{x+\mu} T_\mu^{-1}, \tag{3.3.7}$$

where $\mu \in X_{\text{dom}}$ is chosen such that $x+\mu$ and μ are both dominant. It is easy to verify that $\bar{T}_x$ does not depend on the choice of μ. We see $\bar{T}_x = q^{\frac{1}{2}\widetilde{L}(t_x)} \theta_x$, where $\widetilde{L}(t_x)$ is a function defined by Lusztig in [43, Section 3.1]. Define $\bar{T}_w = T_{w_{\text{fin}}} \bar{T}_x$ for $w = w_{\text{fin}} t_x \in W(\mathcal{R})$, we know from 3.3.4 that $\{\bar{T}_w \mid w \in W(\mathcal{R})\}$ form a basis of $\mathcal{H}$.

Recall our notation for the finite Hecke algebra $H = H(W, q_s)$ and the affine Hecke algebra $\mathcal{H} = \mathcal{H}(W(\mathcal{R}), q_s)$. We now introduce more notions. For a subset $I \subset S$, let W_I denote the subgroup of W generated by I. We define H_I as the subalgebra of the finite Hecke algebra generated by T_w for $w \in W_I$. Let us define $\mathcal{H}_\varnothing = \sum_{\mu \in X} K \cdot \bar{T}_\mu$ and $\mathcal{H}_I = H_I \otimes \mathcal{H}_\varnothing$ which is isomorphic to the affine Hecke algebra associated with $W(\mathcal{R})_I = W_I \ltimes X$. Following [37, Theorem 3.3], we have the following decomposition theorem for T_w with $w \in W(\mathcal{R})$:

Theorem 3.3.6. *The algebra $\mathcal{H}$ in Definition 3.3.3 is generated by T_ρ, for $\rho \in \Omega$ and T_s, for $s \in S_{\text{aff}}$. We write an element w of the affine Weyl group $W(\mathcal{R})$ as $w := \gamma\tau$, $\gamma = w_\Omega t_\mu \in \Omega$ with $w_\Omega \in W$, $\mu \in X$, $\tau \in W_{\text{aff}}$, and $\tau = s_{i_1} \cdots s_{i_r}$, $s_{i_k} \in S_{\text{aff}}$ be a reduced expression of τ. Then*

$$T_w = T_\gamma T_\tau = T_\gamma T_{s_{i_1}} T_{s_{i_2}} \cdots T_{s_{i_r}}.$$

The restriction of an $\mathcal{H}$-module M to $\mathcal{H}_I$ is just the restriction in the usual sense, denoted by $\operatorname{Res}_I M$.

Now we introduce the induced module $(\pi = \operatorname{Ind}_I \sigma, \operatorname{Ind}_I N)$ for an $\mathcal{H}_I$-module (σ, N).

Definition 3.3.7. The induced module $(\pi = \operatorname{Ind}_I \sigma, \operatorname{Ind}_I N)$ (often appears as $\mathcal{H} \otimes_{\mathcal{H}_I} N$) is the module with space $\left(\mathcal{H} \otimes_{\mathcal{H}_\varnothing} N\right) / K(\sigma)$ where $K(\sigma)$ is the subspace generated by

$$\langle hT_s \otimes n - h \otimes \sigma(T_s) n;\ h \in \mathcal{H},\ s \in I,\ n \in N \rangle,$$

and the Hecke algebra action given by

$$\pi(h_1)[h_2 \otimes n + K(\sigma)] = h_1 h_2 \otimes n + K(\sigma),\ h_1, h_2 \in \mathcal{H}.$$

3.4 An example: the Iwahori-Hecke algebra

In this section we will recall an important example of affine Hecke algebra. Recall the notation and conventions for local fields in Section 3.1. Let $\mathbb{G}$ be a split reductive group over a non-Archimedean field F. We denote the p-adic groups $\mathbb{G}(F)$ by G (similar notations for other groups) and choose a maximal split torus $\mathbb{T}$. We fix a Borel subgroup $\mathbb{B} = \mathbb{T}\mathbb{N}$. Then $K = \mathbb{G}(\mathfrak{o})$ is a maximal compact subgroup. By Iwasawa decomposition we have $G = KTN$. Let K_1 denote the unipotent radical of K, then $K/K_1 \cong \mathbb{G}(\mathbb{F}_q)$.

Fix a Borel subgroup of $\mathbb{G}(\mathbb{F}_p)$, then its inverse image is an open compact subgroup called *Iwahori subgroup*, denoted by I. We normalize the Haar measure μ such that $\mu(I) = 1$.

The algebra $\mathcal{H}(I\backslash G/I)$ consisting of locally constant compactly supported bi-invariant functions under convolution is called *Iwahori Hecke algebra.* (If we use notations from Definition 6.4.1, this algebra is denoted by $\mathcal{H}(G, 1_I)$ where 1_I denotes the trivial representation of I.) We know from [18, 4.3]:

Proposition 3.4.1. *The functor $(\pi, V) \mapsto V^I$ is an equivalence between the following categories:*

(1) equivalence classes of (admissible) irreducible smooth finite length representations (π, V) of G such that $V^I \neq 0$.

(2) isomorphism classes of simple finite $\mathcal{H}(I\backslash G/I)$-modules.

Besides, we have $\mathcal{H}(I\backslash G/I) \cong \mathrm{End}_G(\mathrm{ind}_I^G(1_I))$. The generator relation was studied in [16], so this Hecke algebra is also often called *Borel-Iwahori Hecke algebra.*

We also know the structure of this algebra well. The choice of $B = TN$ determines a set of simple (co)roots (*resp.* $\Delta^\vee$) Δ, positive (co)root (*resp.* $\Pi^\vee$) Π in (*resp.* $Y := \mathrm{Hom}(F^\times, T)$) $X := \mathrm{Hom}(T, F^\times)$. Define $Y^+ := \{y \in Y \mid \alpha(y) \geqslant 0, \text{ for all } \alpha \in \Pi\}$. Let W denote the finite Weyl group given by $N_G(T)/T$, and s_α the reflection corresponding to the simple roots. For $w \in W$, set

$$T_w := \text{Characteristic function of double cosets } I n_w I.$$

where n_w is a representative of w in $N_G(T)$. We have the following quadratic equation that T_{s_α} satisfies

$$(T_{s_\alpha} + 1)(T_{s_\alpha} - q_{s_\alpha}) = 0.$$

where $q_{s_\alpha} = \mathrm{Card}(Is_\alpha I/I)$. For $y \in Y^+$, set

$$\theta_y := q^{-\frac{1}{2}\sum_{\alpha\in\Pi}\alpha(y)} \times \text{Characteristic function of double cosets } Iy(\varpi_F)I.$$

For any $y \in Y$, write it as: $y = y_1 - y_2$ for y_1, $y_2 \in Y^+$, define $\theta_y := \theta_{y_1}\theta_{y_2}^{-1}$. Writing in the form compatible with Definition 3.3.4, we have a decomposition

$$\mathcal{H}(I\backslash G/I) \cong \mathcal{H}(I\backslash K/I) \otimes_{\mathbb{C}} A \cong \mathcal{H}(\mathbb{B}(\mathbb{F}_q)\backslash\mathbb{G}(\mathbb{F}_q)/\mathbb{B}(\mathbb{F}_q)) \otimes_{\mathbb{C}} A.$$

In the above decomposition, A is the algebra generated by θ_L for $L \in \mathrm{Hom}(F^\times, T)$, and $\mathcal{H}(I\backslash K/I)$ has a basis given by T_w, with $w \in W$.

Chapter 4

The Involution theorem

We use the notations from previous Chapter. Recall that $H = H(W, q_s)$ is the finite Hecke algebra associated with the Coxeter system (W, S), and $\mathcal{H} = \mathcal{H}(W(\mathcal{R}), q_s)$ is the generalized affine Hecke algebra defined in Definition 3.3.3 with not necessarily equal parameters. We make the assumption that the root system R is irreducible. Let $\mathfrak{R}(\mathcal{H})$ be the Grothendieck group of finite dimensional $\mathcal{H}$-modules over K. We define the dual of an $\mathcal{H}$-module (π, M) (we omit π when no confusion can be caused) by

$$D[M] = \sum_{I \subset S} (-1)^{|I|} [\mathrm{Ind}_I(\mathrm{Res}_I M)]. \tag{4.0.1}$$

Define

$$T_w^* := (-1)^{\ell(w_\Omega)} \prod_{k=1}^{r} (-q_{s_{i_k}}) T_{w^{-1}}^{-1}, \tag{4.0.2}$$

where notations follow Theorem 3.3.6 in the decomposition of T_w. For an $\mathcal{H}$-module (π, M), denote (π^*, M^*) the $\mathcal{H}$-module obtained by twisting the action of $\mathcal{H}$ by $*$. To be more precise, the module $M^* = M$ as K-vector space, equipped with the twisted action $\pi^*(h)(m) := \pi(h^*)(m)$ for $\forall h \in \mathcal{H}$ and $\forall m \in M$. We define

$$q(w) = \prod_{k=1}^{r} q_{s_{i_k}}.$$

We recall notations here: $w = w_{\mathrm{fin}} t_x$ is the decomposition for w as an element in $W \ltimes X$; and $w = \gamma\tau$ is the decomposition for w as an element in $\Omega \ltimes W_{\mathrm{aff}}$, with $\gamma \in \Omega$ with length 0 and τ a product of $\ell(w)$ elements in S_{aff}. Moreover, $\gamma \in \Omega$ can further be written as $w_\Omega t_\mu$, $w_\Omega \in W$, $t_\mu \in X$ and $\tau = w_\Omega t_\mu s_{i_1} s_{i_2} \cdots s_{i_r}$ where $r = \ell(w) = \ell(\tau)$. The above defined involution on Hecke algebras can be rewritten as

$$T_w^* = (-1)^{\ell(w_\Omega) + \ell(w)} q(w) T_{w^{-1}}^{-1}. \tag{4.0.3}$$

Proposition 4.0.1. *We have* $T_w^* = (-1)^{\ell(w_{\text{fin}})} q(w) T_{w^{-1}}^{-1}$ *for*

$$w = w_{\text{fin}} t_x = \gamma\tau = w_\Omega t_\mu s_{i_1} s_{i_2} \cdots s_{i_r}.$$

Proof. We need to show that $\ell(w_\Omega) + \ell(w) \equiv \ell(w_{\text{fin}}) \mod 2$. Notice that $w = w_\Omega t_\mu \tau = w_{\text{fin}} t_x$. Among all simple reflections appearing in the decomposition $\tau = s_{i_1} s_{i_2} \cdots s_{i_r}$, some of them are $s_0 = s_{\alpha_0} t_{-\alpha_0}$, assume that there are N such places. To illustrate the idea, we specify one such term with largest index (k-th place) by writing

$$\tau = s_{i_1} s_{i_2} \cdots s_{i_r} = s_{i_1} s_{i_2} \cdots s_{i_{k-1}} (s_{\alpha_0} t_{-\alpha_0}) s_{i_{k+1}} \cdots s_{i_r}.$$

We also decompose w_Ω into a product of $\ell(w_\Omega) = n$ elements: $w_\Omega = s_{j_1} s_{j_2} \cdots s_{j_n}$. Put these into the expression of w, we get

$$w = s_{j_1} s_{j_2} \cdots s_{j_n} t_\mu s_{i_1} s_{i_2} \cdots s_{i_{k-1}} (s_{\alpha_0} t_{-\alpha_0}) s_{i_{k+1}} \cdots s_{i_r}. \tag{4.0.4}$$

Notice that for any element $w_0 \in W$, $\lambda_0 \in X$, we always have the relation $w_0 t_{\lambda_0} w_0^{-1} = t_{w_0(\lambda_0)}$. Thus $t_{-\alpha_0} s_{i_{k+1}} \cdots s_{i_r} = s_{i_{k+1}} \cdots s_{i_r} t_{-(s_{i_{k+1}} \cdots s_{i_r})^{-1}\alpha_0}$. Put this into the expression for w, we get

$$w = w' t_{(s_{i_k} \cdots s_{i_r})^{-1}\alpha_0}, \quad w' = s_{j_1} s_{j_2} \cdots s_{j_n} t_\mu s_{i_1} s_{i_2} \cdots s_{i_{k-1}} s_{\alpha_0} s_{i_{k+1}} \cdots s_{i_r}. \tag{4.0.5}$$

Repeat this process in w' for the all places where s_0 appear to put translations to the right side, starting from large subindexes to small subindexes and do it also for t_μ. We finally write w as an element in the finite Weyl group mulitiplying a translation:

$$w = s_{j_1} s_{j_2} \cdots s_{j_n} s_{i_1} s_{i_2} \cdots s_{i_{k-1}} s_{\alpha_0} s_{i_{k+1}} \cdots s_{i_r} t_\phi, \ \phi \in X. \tag{4.0.6}$$

By the uniqueness of the expression for $w = w_{\text{fin}} t_x$, we have

$$w_{\text{fin}} = s_{j_1} s_{j_2} \cdots s_{j_n} s_{i_1} s_{i_2} \cdots s_{i_{k-1}} s_{\alpha_0} s_{i_{k+1}} \cdots s_{i_r}, \tag{4.0.7}$$

as $n + r$ elements product. We will need a few facts:

(1) The length $\ell(s_{\alpha_0})$ is odd integer because it is a reflection.

(2) By the *deletion condition* for general Coxeter system (see [36, Section 5.8 Corollary]), we know in obtaining the reduced expression from a not necessary reduced one, simple reflections are cancelled by pairs. Thus for any element of the form $\prod_i u_i$ in a general Coxeter system, $\ell(\prod_i u_i) \equiv \sum_i \ell(u_i) \mod 2$.

(3) Recall that $\ell(w_\Omega) = n$, $\ell(w) = \ell(\tau) = r$.

We have

$$\begin{aligned}\ell(w_{\text{fin}}) \equiv& \ell(w_\Omega) + \sum_{j=0}^{N} \ell(s_{i_{j,1}} \cdots s_{i_{j,k_j-1}}) + N\ell(s_{\alpha_0}) \\ \equiv& n + (\sum_{j=0}^{N} \ell(s_{i_{j,1}} \cdots s_{i_{j,k_j-1}}) + N) \\ \equiv& n + \ell(s_{i_1} s_{i_2} \cdots s_{i_{k-1}} s_0 s_{i_{k+1}} \cdots s_{i_r}) \\ =& n + r = \ell(w_\Omega) + \ell(w) \mod 2,\end{aligned} \tag{4.0.8}$$

where k_j, $j = 1, \ldots, N$ are the places for s_{α_0} to appear in (4.0.7), $s_{i_{j,1}} \cdots s_{i_{j,k_j-1}}$ is the product of simple reflections in W between the j-th and $j+1$-th s_{α_0}.

□

The main goal of this Chapter is to prove the following:

Theorem 4.0.2. *We have $D[M] = [M^*]$ for any $\mathcal{H}$-module M, where $*$ is the involution on the elements of $\mathcal{H}$ defined by*

$$T_w^* = (-1)^{\ell(w_{\text{fin}})} q(w) T_{w^{-1}}^{-1}.$$

4.1 Proof of the main Theorem

Now we start the proof of Theorem 4.0.2 by writing

$$\operatorname{Ind}_I(\operatorname{Res}_I M) = \mathcal{H} \otimes_{\mathcal{H}_\varnothing} M / \langle hT_s \otimes m - h \otimes \pi(T_s)m;\ h \in \mathcal{H},\ s \in I,\ m \in M \rangle .$$

From the Bernstein-Lusztig presentation $\mathcal{H} = H \otimes_K \mathcal{H}_\varnothing$, we have $\mathcal{H} \otimes_{\mathcal{H}_\varnothing} M \cong H \otimes_K M$ as vector spaces. Define $\tau_s \in \operatorname{End}_K(\mathcal{H} \otimes_K M)$ for $s \in S$ by

$$\tau_s(h \otimes m) = hT_s \otimes \pi(T_s)^{-1} m - h \otimes m.$$

There is a corresponding element of τ_s in $\operatorname{End}_K(\mathcal{H} \otimes_{\mathcal{H}_\varnothing} M)$, by abuse of notation we still denote it by τ_s. We have

$$\operatorname{Ind}_I(\operatorname{Res}_I M) \cong \mathcal{H} \otimes_{\mathcal{H}_\varnothing} M / \sum_{s \in I} L_s. \quad (L_s = \operatorname{Im} \tau_s)$$

Let us identify $M^W = K^W \otimes_K M$ with $\mathcal{H} \otimes_K M$ by the K-linear map:

$$\varphi : M^W \to \mathcal{H} \otimes_K M \quad \text{defined by } \varphi((m_w)_{w \in W}) = \sum_{w \in W} T_w \otimes \pi(T_w)^{-1} m_w. \tag{4.1.1}$$

Composed with the isomorphism $H \otimes_K M \cong \mathcal{H} \otimes_{\mathcal{H}_\varnothing} M$, we have a corresponding map $M^W \to \mathcal{H} \otimes_{\mathcal{H}_\varnothing} M$ and still denote it by φ.

Lemma 4.1.1. *For $s \in S$, put*

$$K_s^W = \{(x_w)_{w \in W} \in K^W \mid x_{ws} = -x_w,\ w \in W\}$$

Then we have $\varphi^{-1}(L_s) = K_s^W \otimes_K M$.

Proof. Here the computation is done for $H \otimes_K M$, when we apply it we will precompose with $H \otimes_K M \cong \mathcal{H} \otimes_{\mathcal{H}_\varnothing} M$. Let $v \in W$ be an element of W with $\ell(vs) > \ell(v)$. Then for m_v, $m_{vs} \in M$, we have

$$\begin{aligned}\tau_s(T_v \otimes \pi(T_v)^{-1}m_v + T_{vs} \otimes \pi(T_{vs})^{-1}m_{vs}) &= T_{vs} \otimes \pi(T_{vs})^{-1}m_v - T_v \otimes \pi(T_v)^{-1}m_v \\ &\quad + T_{vs}T_s \otimes \pi(T_{vs}T_s)^{-1}m_{vs} - T_{vs} \otimes \pi(T_{vs})^{-1}m_{vs} \\ &= A + B + C + D.\end{aligned}$$

where $A := +T_{vs} \otimes \pi(T_{vs})^{-1}m_v$, $B := -T_v \otimes \pi(T_v)^{-1}m_v$, $C := +T_{vs}T_s \otimes \pi(T_{vs}T_s)^{-1}m_{vs}$, and $D := -T_{vs} \otimes \pi(T_{vs})^{-1}m_{vs}$.

For $s \in S$, we will need the following easy facts:

$$T_s^2 = q_s + (q_s - 1)T_s, \tag{4.1.2}$$

$$T_s = \frac{q_s - T_s^2}{1 - q_s}, \tag{4.1.3}$$

$$T_s^{-1} = q_s^{-1}T_s - (1 - q_s^{-1}). \tag{4.1.4}$$

We have

$$\begin{aligned}C + D &= T_vT_s^2 \otimes \pi(T_vT_s^2)^{-1}m_{vs} - T_{vs} \otimes \pi(T_{vs})^{-1}m_{vs} \\ &= T_v(q_s + (q_s - 1)T_s) \otimes \pi(T_vT_s^2)^{-1}m_{vs} - T_{vs} \otimes \pi(T_{vs})^{-1}m_{vs} \text{ (Using (4.1.2))} \\ &= T_v \otimes q_s\pi(T_v)^{-1}\pi(T_v)\pi(T_vT_s^2)^{-1}m_{vs} - T_{vs} \otimes \pi(T_{vs})^{-1}m_{vs} \\ &+ T_{vs} \otimes (q_s - 1)\pi(T_{vs})^{-1}\pi(T_{vs})\pi(T_vT_s^2)^{-1}m_{vs}\end{aligned}$$

Rewrite the last term:

$$\begin{aligned}&T_{vs} \otimes (q_s - 1)\pi(T_{vs})^{-1}\pi(T_{vs})\pi(T_vT_s^2)^{-1}m_{vs} \\ &\qquad = T_{vs} \otimes \pi(T_{vs})^{-1}(q_s - 1)\pi(T_vT_sT_s^{-2}T_v^{-1})m_{vs} \\ \text{(Using (4.1.3))} &\qquad = T_{vs} \otimes \pi(T_{vs})^{-1}\pi(T_v(T_s^2 - q_s)T_s^{-2}T_v^{-1})m_{vs} \\ &\qquad = T_{vs} \otimes \pi(T_{vs})^{-1}m_{vs} - T_{vs} \otimes \pi(T_{vs})^{-1}q_s\pi(T_vT_s^{-2}T_v^{-1})m_{vs}.\end{aligned}$$

Hence

$$C + D = T_v \otimes \pi(T_v)^{-1}q_s\pi(T_vT_s^{-2}T_v^{-1})m_{vs} - T_{vs} \otimes \pi(T_{vs})^{-1}q_s\pi(T_vT_s^{-2}T_v^{-1})m_{vs}.$$

and

$$A+B+C+D = T_{vs} \otimes \pi(T_{vs})^{-1}m' - T_v \otimes \pi(T_v)^{-1}m', \tag{4.1.5}$$

where $m' = m_v - q_s\pi(T_vT_s^{-2}T_v^{-1})m_{vs}$. This shows that the inverse image has $-m'$, m' repectively for places indexed by v, vs, thus $\varphi^{-1}(L_s) \subset K_s^W \otimes_K M$. If we let $m_{vs} = 0$, then $\varphi^{-1}(L_s)$ has $-m_v$ and m_v respectively for places indexed by v, vs, thus $\varphi^{-1}(L_s) \supset K_s^W \otimes_K M$. □

Thus we have a K-isomorphism

$$(K^W/\sum_{s\in I} K_s^W) \otimes_K M \cong \mathcal{H} \otimes_{\mathcal{H}_\varnothing} M/\sum_{s\in I} L_s.$$

We can further identify $K^W/\sum_{s\in I} K_s^W$ with $K[W/W_I]$ by the map

$$(x_w)_{w\in W} \mapsto \sum_{w\in W} x_w \cdot wW_I.$$

Now we will use the complex introduced by Deligne-Lusztig [27] and S-I. Kato [39]. First, define

$$\pi_I^{I'} : \mathcal{H} \otimes_{\mathcal{H}_\varnothing} M/\sum_{s\in I} L_s \to \mathcal{H} \otimes_{\mathcal{H}_\varnothing} M/\sum_{s\in I'} L_s,$$

as the natural projection for $I \subset I' \subset S$ where I' is obtained by adding one more element from $S\backslash I$. We know that $\pi_I^{I'}$ is an $\mathcal{H}$-homomorphism.

To give the set of simple roots S an ordering with respect to the Weyl group action as in [58, Section 4] and thus making the complex below (4.1.6) well-defined ($d \circ d = 0$), we define:

$$\epsilon_I^{I'} : \bigwedge^{|I|}(K^I) \to \bigwedge^{|I'|}(K^{I'})$$

be the natural isomorphism given by $v \to v \wedge s$, $s \in I'\backslash I$. Here K^X is the free K-modules for any set X, and put $\bigwedge^{|I|}(K^I) = K$ for $I = \varnothing$.

Then we can define the following complex of $\mathcal{H}$-modules:

$$0 \to C_0 \xrightarrow{d_0} C_1 \xrightarrow{d_1} \dots \xrightarrow{d_{i-1}} C_i \xrightarrow{d_i} \dots \xrightarrow{d_{|S|-1}} C_{|S|} \to 0, \tag{4.1.6}$$

where

$$C_i = \bigoplus_{|I|=i} (\mathcal{H} \otimes_{\mathcal{H}_\varnothing} M/\sum_{s\in I} L_s) \otimes \bigwedge^i(K^I),$$

and $d_i = \pi_I^{I'} \otimes \epsilon_I^{I'}$ with $|I| = i$.

Then using the complex by [58] for simplicial decomposition of $(|S|-1)$-dimensional sphere, or by [27]'s main theorem, we know the complex 4.1.6 has a nonzero cohomology only at the degree 0. Thus

$$\begin{aligned} D[M] &= \sum_i (-1)^i [C_i] = [\mathrm{Ker}\, d_0] \\ &= [\bigcap_{s\in S} L_s] = [\varphi(\bigcap_{s\in S}(K_s^W \otimes_K M))]. \end{aligned} \tag{4.1.7}$$

Any element of $\bigcap_{s\in S}(K_s^W \otimes_K M)$ is of the form $\left((-1)^{\ell(w)} x_1 \otimes m\right)_{w\in W}$ (where $x_1 \in K$) for some $m \in M$. Therefore

$$\begin{aligned} \varphi\left(\left((-1)^{\ell(w)} x_1 \otimes m\right)_{w\in W}\right) &= \sum_{w\in W} T_w \otimes (\pi(T_w)^{-1}(-1)^{\ell(w)} x_1 \otimes m) \\ &= \sum_{w\in W} (-1)^{\ell(w)} T_w \otimes \left(\pi(T_w)^{-1}(1\otimes x_1 m)\right) \\ &= \sum_{w\in W} (-1)^{\ell(w)} (T_w \otimes \pi(T_w)^{-1})(1\otimes(1\otimes x_1 m)) \\ &= \chi(1\otimes x_1 m) \quad (\text{Identify } 1\otimes x_1 m \text{ with } x_1 m), \end{aligned} \tag{4.1.8}$$

where $\chi = \sum_{w\in W}(-1)^{\ell(w)} T_w \otimes T_w^{-1}$, acting by "left multiply $T_w \otimes \pi$".

We summarize what we have got so far: $D[M] = [\varphi\left(\bigcap_{s\in S}(K_s^W \otimes_K M)\right)] = \chi(1 \otimes_{\mathcal{H}_\varnothing} M)$ where $1 \otimes_{\mathcal{H}_\varnothing} M \subset \mathcal{H} \otimes_{\mathcal{H}_\varnothing} M$. We know that χ is bijective K-homomorphism when restricted to $1 \otimes_{\mathcal{H}_\varnothing} M$ since φ is bijective on $\bigcap_{s\in S}(K_s^W \otimes_K M)$. We now need to show that $\chi(1\otimes M)$ is isomorphic to M^*, or equivalently:

$$(T_w \otimes 1)\chi = \chi(1\otimes T_w^*) = \chi(1\otimes(-1)^{\ell(w_{\mathrm{fin}})} q(w) T_{w^{-1}}^{-1}).$$

This is proved in the following steps:

(1) We use the Iwahori-Matsumoto presentation of T_w to write

$$T_w \otimes 1 = (T_\gamma \otimes 1)(T_{s_{i_1}} \otimes 1)(T_{s_{i_2}} \otimes 1)\cdots(T_{s_{i_k}} \otimes 1),$$

where $w = \gamma s_{i_1} s_{i_2} \cdots s_{i_k}$ with $\gamma = w_\Omega t_\mu \in \Omega$ and $s_{i_t} \in S_{\mathrm{aff}}$ for any $1 \leqslant t \leqslant k$.

(2) We prove $T_{s_{i_t}} \otimes 1$, $s_{i_t} \neq s_0$ intertwines with χ: $(T_{s_{i_t}} \otimes 1)\chi = \chi(1\otimes(-q_{s_{i_t}} T_{s_{i_t}}^{-1}))$ using the first part of Lemma 4.2.1.

(3) We prove $T_{s_0} \otimes 1$ intertwines with χ: $(T_{s_0} \otimes 1)\chi = \chi(1\otimes(-q_{s_0} T_{s_0}^{-1}))$ using the second part of Lemma 4.2.1.

(4) We prove $T_\gamma \otimes 1$ intertwines with χ: $(T_\gamma \otimes 1)\chi = (-1)^{\ell(w_\Omega)}\chi(1\otimes T_\gamma)$ using [39, Lemma 3]..

(5) We conclude that

$$\begin{aligned}(T_w \otimes 1)\chi =&(T_\gamma \otimes 1)(T_{s_{i_1}} \otimes 1)(T_{s_{i_2}} \otimes 1)\cdots(T_{s_{i_k}} \otimes 1)\chi\\ =&\chi((-1)^{l(w_\Omega)}(1 \otimes T_\gamma))(1 \otimes (-q_{s_{i_1}} T_{s_{i_1}}^{-1}))(1 \otimes (-q_{s_{i_2}} T_{s_{i_2}}^{-1}))\cdots(1 \otimes (-q_{s_{i_k}} T_{s_{i_k}}^{-1}))\\ =&\chi(1 \otimes (-1)^{\ell(w_\Omega)+l(w)}(\prod q_{s_i})T_{w^{-1}}^{-1})\\ =&\chi(1 \otimes (-1)^{\ell(w_{\mathrm{fin}})} q(w) T_{w^{-1}}^{-1})\end{aligned} \tag{4.1.9}$$

by proposition 4.0.1.

From the above proof we can claim:

Proposition 4.1.2. *The* $*$ *operation on* $\mathcal{H}$ *is an involution (*i.e. *an automorphism of such that* $* \circ * = \mathrm{Id}$*).*

4.2 A Key Lemma

We recall those definitions introduced in Chapter 3 and along the way introduce more notions. Let X and Y be lattices of finite rank, with a perfect pairing $\langle \cdot, \cdot \rangle \to \mathbb{Z}$. Let R be a root system in X and $R^\vee$ is the coroot (dual root) system such that $\langle \alpha, \alpha^\vee \rangle = 2$. For every $\alpha \in R$, $x \in X$, $s_\alpha(x) := x - \langle x, \alpha^\vee \rangle \alpha$ acts on affine space $E := X \otimes \mathbb{R}$ and stabilizes R. The pairing $\langle \cdot, \cdot \rangle$ extends naturally to $X \otimes \mathbb{R}$ and $Y \otimes \mathbb{R}$, and it can also be viewed as a positive definite inner product on V (the underlying $\mathbb{R}$-vector space of E) via the natural map $\iota : V \to V^\vee$. The finite Weyl group $W = W(R)$ is the group generated by s_α for $\alpha \in R$, thus W can be naturally identified with a subgroup of $\mathrm{GL}(V)$ acting on E on the left. This is called the *ordinary left action.*

Following [38, Section 1.6] and [41, Section 1.2], we introduce a *right* action of $W(\mathcal{R})$ on E: for $v \in E$, $w \in W$ and $\lambda \in X$, the extended affine Weyl group $W(\mathcal{R})$ acts on E on the right as follows:

$$vw := w^{-1} \cdot v, \tag{4.2.1}$$

where $w^{-1} \cdot v$ is the ordinary left action, and

$$vt_\lambda := v + \lambda. \tag{4.2.2}$$

Recall that $\alpha_0^\vee$ is the unique maximal coroot and $s_0 = s_{\alpha_0} t_{-\alpha_0}$, a hyperplane indexed by $\alpha \in R^+$ and $n \in \mathbb{Z}$ is defined as:

$$H_{\alpha,n} = \{x \in E \mid \langle x, \alpha^\vee \rangle = n\}.$$

With respect to the right action, s_0 corresponds to the affine reflection associated with $H_{\alpha_0,-1}$. We denote the set of all connected components of $E \backslash (\bigcup_{\alpha \in R^+, n \in \mathbb{Z}} H_{\alpha,n})$ by $\mathcal{A}$.

We define A^- as the open alcove in E bounded by $H_{\alpha,0}$ for all $\alpha \in R^+$ and $H_{\alpha_0,-1}$. Then $A^- \in \mathcal{A}$, and $\mathcal{A} = \{A^-w \mid w \in W_{\text{aff}}\}$. There is a left action of W_{aff} on $\mathcal{A}$ by $y(A^-w) = A^-yw$ for y, $w \in W_{\text{aff}}$. Let C^+ denote *the dominant cone* (*i.e.* the cone bounded by $H_{\alpha,0}$ for $\alpha \in R$ and containing A^-w_0). For any $A \in \mathcal{A}$, we define $\mathcal{L}(A)$ by

$$\mathcal{L}(A) = \{s \in S_{\text{aff}} \mid A \subset E^+_{H_s}\}, \tag{4.2.3}$$

where H_s is the affine hyperplane associated with s, and $E^+_{H_s}$ is the complement of $E - H_s$ that has nonempty intersection with C^+t_λ, for all $\lambda \in X$.

In the rest part of this section, we correct some steps in S-I. Kato's proof of Lemma 2 and give more details in Lemma 4.2.1. The Proof in [39, Lemma 2] for $(T_{s_0} \otimes 1)\chi = \chi(1 \otimes (-q_{s_0}T_{s_0}^{-1}))$ contains an error when he uses the lemma from [38]: $s \notin \mathcal{L}(A^-w)$ is not equivalent to $y^{-1}(\alpha_0) > 0$ (using S-I. Kato's notation here) in general.

Lemma 4.2.1 (S-I. Kato Lemma 2)**.** *We use the definition of a subset $\mathcal{L}(A^-v)$ of S_{aff} explained as above. Then we have*

$$(T_s \otimes 1)\chi = \chi(1 \otimes (-q_sT_s^{-1}))$$

holds in $\mathcal{H} \otimes_{\mathcal{H}_\varnothing} \mathcal{H}$ for $s \in S_{\text{aff}} = S \cup \{s_0\}$. (We assume the R is irreducible.) Take $v \in W$ with $s \notin \mathcal{L}(A^-v)$, we need to show:

$$(T_s \otimes 1)(T_v \otimes T_v^{-1} - T_{sv} \otimes T_{sv}^{-1}) = (T_v \otimes T_v^{-1} - T_{sv} \otimes T_{sv}^{-1})(1 \otimes (-q_sT_s^{-1})), \text{ for } s \in S \tag{4.2.4}$$

$$(T_{s_0} \otimes 1)(T_v \otimes T_v^{-1} - T_{s_{\alpha_0}v} \otimes T_{s_{\alpha_0}v}^{-1}) = (T_v \otimes T_v^{-1} - T_{s_{\alpha_0}v} \otimes T_{s_{\alpha_0}v}^{-1})(1 \otimes (-q_{s_0}T_{s_0}^{-1})) \tag{4.2.5}$$

Before coming to the proof of Lemma 4.2.1, we need the following two Lemmas from [38, 1.9] and [37]:

Lemma 4.2.2. *For $s \in S_{\text{aff}}$ and $v \in W$, we have*

$$T_sT_v = \begin{cases} \bar{T}_{sv}, & s \notin \mathcal{L}(A^-v) \\ q_s\bar{T}_{sv} + (q_s - 1)T_v, & s \in \mathcal{L}(A^-v). \end{cases} \tag{4.2.6}$$

For $s \in S$, $s \notin \mathcal{L}(A^-v)$ is equivalent to $\ell(sv) = \ell(s) + \ell(v)$ and $\bar{T}_{sv} = T_{sv}$.

Lemma 4.2.3. *(1) For $s_0 = s_{\alpha_0}t_{-\alpha_0} = t_{\alpha_0}s_{\alpha_0}$, we have $T_{\alpha_0} = T_{s_0}T_{s_{\alpha_0}}$.*
(2) We have $T_{t_{v(\lambda)}v} = T_vT_\lambda$ for $\lambda \in X_{\text{dom}}$, $v \in W$.

Proof of the Lemma 4.2.1. Calculate left hands side of (4.2.4):

$$\begin{aligned}(T_s \otimes 1)(T_v \otimes T_v^{-1} - T_{sv} \otimes T_{sv}^{-1}) &= T_sT_v \otimes T_v^{-1} - T_sT_{sv} \otimes T_{sv}^{-1} \\ &= T_{sv} \otimes T_v^{-1} - (q_sT_v + (q_s - 1)T_{sv}) \otimes T_{sv}^{-1} \\ &= -q_sT_v \otimes T_{sv}^{-1} - (q_s - 1)T_{sv} \otimes T_{sv}^{-1} + T_{sv} \otimes T_v^{-1}.\end{aligned}$$

Look at the last two terms:

$$
\begin{aligned}
-(q_s-1)T_{sv}\otimes T_{sv}^{-1}+T_{sv}\otimes T_v^{-1} &= -(q_s-1)T_{sv}\otimes T_v^{-1}(q_s^{-1}T_s-(1-q_s^{-1}))+T_{sv}\otimes T_v^{-1}\\
&= \frac{q_s^2-q_s+1}{q_s}T_{sv}\otimes T_v^{-1}+\frac{1-q_s}{q_s}T_{sv}\otimes T_v^{-1}T_s.
\end{aligned}
$$

Calculate right hands side of (4.2.4):

$$
(T_v\otimes T_v^{-1}-T_{sv}\otimes T_{sv}^{-1})(1\otimes(-q_sT_s^{-1}))=-q_sT_v\otimes T_v^{-1}T_s^{-1}+q_sT_{sv}\otimes T_{sv}^{-1}T_s^{-1}.
$$

Look at the second term above:

$$
\begin{aligned}
q_sT_{sv}\otimes T_{sv}^{-1}T_s^{-1} &= q_sT_{sv}\otimes T_v^{-1}(T_s^{-1})^2\\
\text{(Using (4.1.4))} &= q_sT_{sv}\otimes T_v^{-1}(q_s^{-1}T_s-(1-q_s^{-1}))^2\\
\text{(Using (4.1.2))} &= q_sT_{sv}\otimes T_v^{-1}(q_s^{-2}(q_s+(q_s-1)T_s)+(1-q_s^{-1})^2-2q_s^{-1}(1-q_s^{-1})T_s)\\
&= \frac{q_s^2-q_s+1}{q_s}T_{sv}\otimes T_v^{-1}+\frac{1-q_s}{q_s}T_{sv}\otimes T_v^{-1}T_s.
\end{aligned}
$$

Thus this finishes the verification of (4.2.4). Now check for (4.2.5).

We can assume $s_0\notin\mathcal{L}(A^-v)$, then $s_0\in\mathcal{L}(A^-s_{\alpha_0}v)$ and $T_{s_0}T_v=\bar{T}_{s_0v}$ by (4.2.6), we have

$$
\bar{T}_{s_0v}=\bar{T}_{t_{\alpha_0}s_{\alpha_0}v}=\bar{T}_{s_{\alpha_0}vt_{-v^{-1}(\alpha_0)}}=T_{s_{\alpha_0}v}\bar{T}_{-v^{-1}(\alpha_0)}.
$$

On the other hand, by comparing the length we have $T_{s_0}T_v=T_{s_0v}$, thus

$$
\begin{aligned}
T_{s_{\alpha_0}v}\otimes T_{s_{\alpha_0}v}^{-1} &= T_{s_{\alpha_0}v}\bar{T}_{-v^{-1}(\alpha_0)}\otimes\bar{T}_{-v^{-1}(\alpha_0)}^{-1}T_{s_{\alpha_0}v}^{-1}\\
&= \bar{T}_{s_0v}\otimes\bar{T}_{s_0v}^{-1}=T_{s_0v}\otimes T_{s_0v}^{-1}.
\end{aligned}
\tag{4.2.7}
$$

We see the (4.2.5) becomes

$$
(T_{s_0}\otimes 1)(T_v\otimes T_v^{-1}-T_{s_0v}\otimes T_{s_0v}^{-1})=(T_v\otimes T_v^{-1}-T_{s_0v}\otimes T_{s_0v}^{-1})(1\otimes(-q_{s_0}T_{s_0}^{-1})),\tag{4.2.8}
$$

whose verification is the same as (4.2.4). □

Following [39, Lemma 3], we have

Lemma 4.2.4 (S-I. Kato Lemma 3). *For $\gamma\in\Omega$ with $\gamma=w_\Omega t_\mu$, in $\mathcal{H}\otimes_{\mathcal{H}_\varnothing}\mathcal{H}$*

$$
(T_\gamma\otimes 1)\chi=(-1)^{\ell(w_\Omega)}\chi(1\otimes T_\gamma).\tag{4.2.9}
$$

4.3 The involution using the Bernstein-Lusztig presentation

We know that $\mathcal{H}(W(\mathcal{R}), q_s)$, according to Theorem 3.3.5, has a basis consisting of $\theta_x T_w$ for $x \in X$ and $w \in W$. By applying the involution in Theorem 4.0.2 to θ_x in the Bernstein-Lusztig presentation, we can deduce the involution in this setting. This version and its proof are first seen in [22]. We follow their approach, even if we do not restrict to the equal parameter case, the formula and the proof does not change much.

Proposition 4.3.1. *In the Bernstein-Lusztig presentation, the involution $*$ in Theorem 4.0.2 acts as*

$$\theta_x^* = T_{w_0}\theta_{w_0(x)}T_{w_0}^{-1}$$

and

$$T_w^* = (-1)^{\ell(w)}q(w)T_{w^{-1}}^{-1},$$

where w_0 is the longest element of the finite Weyl group W.

Proof. Recall that $\ell(w_0) = \mathrm{Card}(R^+)$, for any $x \in X_{\mathrm{dom}}$ by (3.2.1), we have:

$$\ell(w_0 t_x) = \sum_{\alpha\in R^+}(1 + \langle x, \alpha^\vee\rangle) = \ell(w_0) + \ell(t_x) = \ell(t_{w_0(x)}) + \ell(w_0).$$

We have $T_{w_0(x)}T_{w_0} = T_{w_0}T_x$, thus

$$\theta_x = q(x)^{-1/2}T_x = q(x)^{-1/2}T_{w_0}^{-1}T_{w_0(x)}T_{w_0}. \tag{4.3.1}$$

Apply the involution to both sides of the above equation:

$$\begin{aligned} \theta_x^* =& q(x)^{-1/2}q(w_0(x))T_{w_0}T_{-w_0(x)}^{-1}T_{w_0}^{-1} \\ =& q(x)^{-1/2}q(w_0(x))^{1/2}T_{w_0}\theta_{w_0(x)}T_{w_0}^{-1} \\ =& T_{w_0}\theta_{w_0(x)}T_{w_0}^{-1}, \end{aligned} \tag{4.3.2}$$

where we used $q(t_{w_0(x)}) = q(w_0 t_x w_0^{-1}) = q(x)$ to get the last equation. □

4.4 Unitarity of the involution

Definition 4.4.1. Let A be a $\mathbb{C}$-algebra. A star operation on A is a ring anti-involution $\kappa : A \to A$, such that for $a, b \in A$

1. $\kappa(a + b) = \kappa(a) + \kappa(b)$,
2. $\kappa(ab) = \kappa(b)\kappa(a), a, b \in A$,
3. κ is conjugate-linear, $\kappa(\lambda a) = \bar{\lambda}\kappa(a), \lambda \in \mathbb{C}, a \in A$.

Definition 4.4.2. Let A be a $\mathbb{C}$-algebra with a *star operation* κ and let M be a Hilbert space carries an A-module structure. We say that M is κ-unitary if M has a positive-definite inner product $\langle\ ,\ \rangle_M$ which is κ-invariant, *i.e.*:

$$\langle a \cdot m_1, m_2 \rangle_M = \langle m_1, \kappa(a) \cdot m_2 \rangle_M, \quad \text{for all } a \in A, m_1, m_2 \in M. \tag{4.4.1}$$

For an affine Hecke algebra $\mathcal{H}$ define over $\mathbb{C}$, there is a natural star operation. In the Iwahori-Matsumoto presentation it is simply $\kappa(T_w) = T_{w^{-1}}$ for all $w \in W(\mathcal{R})$. Let us recall the involution we define earlier $T_w^* = (-1)^{\ell(w_{\text{fin}})} q(w) T_{w^{-1}}^{-1}$ for $w = w_{\text{fin}} t_x = \gamma\tau = w_\Omega t_\mu s_{i_1} s_{i_2} \cdots s_{i_r}$.

Lemma 4.4.3. *Under the above notations, we have* $\langle T_{w^{-1}}^{-1} \cdot m_1, m_2 \rangle_M = \langle m_1, T_w^{-1} \cdot m_2 \rangle_M$.

Proof. Let $m_1 = T_{w^{-1}} \cdot m_1'$, we have

$$\begin{aligned} \langle m_1, T_w^{-1} \cdot m_2 \rangle_M =& \langle T_{w^{-1}} \cdot m_1', T_w^{-1} \cdot m_2 \rangle_M \\ =& \langle m_1', T_w T_w^{-1} \cdot m_2 \rangle_M = \langle m_1', m_2 \rangle_M \\ =& \langle T_{w^{-1}}^{-1} T_{w^{-1}} \cdot m_1', m_2 \rangle_M = \langle T_{w^{-1}}^{-1} m_1, m_2 \rangle_M. \end{aligned} \tag{4.4.2}$$

□

Theorem 4.4.4. *If the values of q_s take real values, the involution $(\cdot)^*$ preserves unitarity.*

Proof. From the definition, the underlying space of M^* is automatically a Hilbert space and $\langle \cdot\ ,\ \cdot \rangle_{M^*}$ is positive-definite. We need to show $\langle T_w \cdot m_1, m_2 \rangle_{M^*} = \langle m_1, \kappa(T_w) \cdot m_2 \rangle_{M^*}$ which is just $\langle T_w^* \cdot m_1, m_2 \rangle_M = \langle m_1, \kappa(T_w)^* \cdot m_2 \rangle_M$. We have

$$\begin{aligned} \langle m_1, \kappa(T_w)^* \cdot m_2 \rangle_M &= \langle m_1, (-1)^{\ell((w^{-1})_{\text{fin}})} q(w^{-1}) T_w^{-1} \cdot m_2 \rangle_M \\ &= \langle m_1, (-1)^{\ell(w_{\text{fin}})} q(w) T_w^{-1} \cdot m_2 \rangle_M \\ &= \langle (-1)^{\ell(w_{\text{fin}})} q(w) T_{w^{-1}}^{-1} m_1, \cdot m_2 \rangle_M = \langle T_w^* \cdot m_1, m_2 \rangle_M. \end{aligned} \tag{4.4.3}$$

We have used the fact $\ell(w_{\text{fin}}) = \ell((w^{-1})_{\text{fin}})$ in the second equation, and used Lemma 4.4.3 in the third equation above. □

The Iwahori-Hecke algebra in Section 3.4 satisfy that $q_s \in \mathbb{R}$, the involution thus preserves the unitarity. D. Barbasch and A. Moy in [11] proved the following theorem:

Theorem 4.4.5. *Let $G = \mathbb{G}(F)$ be a split reductive group with connected center over F, an irreducible module with nonzero Iwahori fixed vector is unitary if and only if the corresponding (via the functor in 3.4.1) module of the Iwahori Hecke algebra is unitary.*

This shows that the counterpart of Hecke algebra involution on the group side, which we will discuss in Section 6.2, preserves unitarity when restricted to certain subcategory.

Chapter 5

A relative version of the involution via the Howlett-Lehrer theory

5.1 Notation and conventions

p: a prime number.
$k := \mathbb{F}_q$ with $q = p^n$ for some integer $n \geqslant 1$.
$\bar{k} = \overline{\mathbb{F}_p}$: a fixed algebraic closure of k.
A: a commutative ring with unit such that p is invertible in A.
Let H be a finite group, then we define:
AH- mod: the category of finite dimensional AH modules.
$\mathbf{G}$: a connected reductive algebraic groups defined over $\bar{k}$, together with an endomorphism F, a power of which is a Frobenius endomorphism.
$\mathrm{Irr}_A(\mathbf{G}^{\mathrm{F}})$: the set of simple $A\mathbf{G}^{\mathrm{F}}$-modules up to isomorphism (also denoted by $\mathrm{Irr}(\mathbf{G}^{\mathrm{F}})$ if no confusion is caused).
We keep the notations for Weyl groups as in section 3.2. We also need the following convention:
${}^gH := gHg^{-1}$, and $H^g := g^{-1}Hg$ where g is an element of group G and H is a set with left-right G action. For any representation π of a group G, let g be an element of G, define a representation ${}^g\pi$ of ${}^gG$ by transport of structure ${}^g\pi(x) := \pi(g^{-1}xg)$ for any $x \in {}^gG$.

5.2 A review of the Harish-Chandra theory

This subsection is mainly based on [28, Chapter 5] and [29]. Recall that $\mathbf{G}$ is a connected reductive group over $\mathbb{F}_q$. We denote by $G = \mathbf{G}^{\mathrm{F}}$ the finite group of fixed points. Let $\mathbf{P} = \mathbf{L}\mathbf{U}$ be a F-stable parabolic subgroup of $\mathbf{G}$ with F-stable Levi complement $\mathbf{L}$. The group algebra $A\mathbf{G}^F/\mathbf{U}^F$ over A is viewed as a $(A\mathbf{G}^F, A\mathbf{L}^F)$-bimodule where $\mathbf{G}^F$ acts by left translations and $\mathbf{L}^F$ by right translations.

Definition 5.2.1. The *Harish-Chandra induction* and *Harish-Chandra restriction* functors are

$$\begin{aligned} R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} : A\mathbf{L}^F\text{-}\mathrm{mod} &\longrightarrow A\mathbf{G}^F\text{-}\mathrm{mod} \\ M &\longmapsto A\mathbf{G}^F/\mathbf{U}^F \otimes_{A\mathbf{L}^F} M, \\ {}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} : A\mathbf{G}^F\text{-}\mathrm{mod} &\longrightarrow A\mathbf{L}^F\text{-}\mathrm{mod} \\ N &\longmapsto \mathrm{Hom}_{A\mathbf{G}^F}\left(A\mathbf{G}^F/\mathbf{U}^F, N\right). \end{aligned}$$

We call an F-stable Levi subgroup $\mathbf{G}$-*split* if it is a Levi subgroup of an F-stable parabolic subgroup of $\mathbf{G}$. We summarize the important properties of Harish-Chandra induction and restriction in the following proposition whose proofs are in [28, Chapter 5]:

Proposition 5.2.2. *(i)* $R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} = \mathrm{Ind}^{A\mathbf{G}^F}_{A\mathbf{P}^F} \circ \mathrm{Inf}^{A\mathbf{P}^F}_{A\mathbf{L}^F}$ *(where the inflation* $\mathrm{Inf}^{A\mathbf{P}^F}_{A\mathbf{L}^F}$ *is the trivial natural lifting through the quotient* $\mathbf{P}^F \to \mathbf{L}^F$ *).*

(ii) ${}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}$ *is* $\mathrm{Res}^{A\mathbf{G}^F}_{A\mathbf{P}^F}$ *followed with the taking of fixed points under* $\mathbf{U}^F$*.*

(iii) $R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}$ *and* ${}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}$ *are exact and biadjoint,* i.e.

$$\begin{aligned} \mathrm{Hom}_{A\mathbf{G}^F}\left(R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(M), N\right) &\cong \mathrm{Hom}_{A\mathbf{L}^F}\left(M, {}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(N)\right) \\ \mathrm{Hom}_{A\mathbf{G}^F}\left(N, R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(M)\right) &\cong \mathrm{Hom}_{A\mathbf{L}^F}\left({}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(N), M\right) \end{aligned} \tag{5.2.1}$$

for $M \in A\mathbf{L}^{\mathrm{F}}\text{-}\mathrm{mod}$ *and* $N \in A\mathbf{L}^{\mathrm{F}}\text{-}\mathrm{mod}$*.*

(iv) (Transitivity) Let $\mathbf{Q}$ *be an* F*-stable parabolic subgroup contained in* $\mathbf{P} = \mathbf{LU}$*, and* $\mathbf{M}$ *an* F*-stable Levi subgroup of* $\mathbf{Q}$ *contained in* $\mathbf{L}$*, then*

$$R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} \circ R^{\mathbf{L}}_{\mathbf{M}\subset\mathbf{L}\cap\mathbf{Q}} = R^{\mathbf{G}}_{\mathbf{M}\subset\mathbf{Q}}, \quad {}^*R^{\mathbf{L}}_{\mathbf{M}\subset\mathbf{Q}\cap\mathbf{L}} \circ {}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} = {}^*R^{\mathbf{G}}_{\mathbf{M}\subset\mathbf{Q}}.$$

(v) (Independence of the parabolic groups) Let $\mathbf{L}$ *be a common* F*-stable Levi component of the* F*-stable parabolic subgroup* $\mathbf{P}$ *and* $\mathbf{P}'$*, then*

$${}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} \cong {}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}'}, \quad R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} \cong R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}'}.$$

We remark that the assumption p is invertible in A is essential for (v) of Proposition 5.2.2. We may omit the parabolic subgroups in the notation for Harish-Chandra induction and restriction in $\mathbf{G}$-split cases.

Theorem 5.2.3 (Mackey formula)**.** *Let* $\mathbf{P}$ *and* $\mathbf{Q}$ *be two* F*-stable parabolic subgroups of* $\mathbf{G}$*, and* $\mathbf{L}$ *(*resp. $\mathbf{M}$*) be an* F*-stable Levi subgroup of* $\mathbf{P}$ *(*resp. $\mathbf{Q}$*). Then*

$${}^*R^{\mathbf{G}}_{\mathbf{L}} \circ R^{\mathbf{G}}_{\mathbf{M}} = \bigoplus_{x \in \mathbf{L}^{\mathrm{F}} \backslash \mathcal{S}(\mathbf{L},\mathbf{M})^{\mathrm{F}} / \mathbf{M}^{\mathrm{F}}} R^{\mathbf{L}}_{\mathbf{L}\cap {}^x\mathbf{M}} \circ {}^*R^{{}^x\mathbf{M}}_{\mathbf{L}\cap {}^x\mathbf{M}} \circ \mathrm{ad}\, x \tag{5.2.2}$$

with $\mathcal{S}(\mathbf{L},\mathbf{M}) = \{x \in \mathbf{G} \mid \mathbf{L} \cap {}^x\mathbf{M}$ *contains a maximal torus of* $\mathbf{G}\}$ *and* $\mathrm{ad}\, x : A\mathbf{M}\text{-}\mathrm{mod} \to A{}^x\mathbf{M}\text{-}\mathrm{mod}$ *denotes the action of* x *by conjugation on the representations.*

Using the same notations as the above theorem, the fact that the inclusion $\mathcal{S}(\mathbf{L},\mathbf{M}) \hookrightarrow \mathbf{G}$ induces a bijection $\mathbf{L}^{\mathrm{F}}\backslash\mathcal{S}(\mathbf{L},\mathbf{M})^{\mathrm{F}}/\mathbf{M}^{\mathrm{F}} \cong \mathbf{P}^{\mathrm{F}}\backslash\mathbf{G}^{\mathrm{F}}/\mathbf{Q}^{\mathrm{F}}$ leads to the following Corollary:

Corollary 5.2.4. *Let I and J be two* F*-stable subsets of S. Then*

$$ {}^*R^{\mathbf{G}}_{\mathbf{L}_I} \circ R^{\mathbf{G}}_{\mathbf{L}_J} \cong \sum_{w \in W_I^{\mathrm{F}}\backslash W^{\mathrm{F}}/W_J^{\mathrm{F}}} R^{\mathbf{L}_I}_{\mathbf{L}_I \cap {}^w\mathbf{L}_J} \circ {}^*R^{{}^w\mathbf{L}_J}_{\mathbf{L}_I \cap {}^w\mathbf{L}_J} \circ \operatorname{ad} w. \tag{5.2.3} $$

Definition 5.2.5. An $A\mathbf{G}^F$-module M is said to be *cuspidal* if ${}^*R^{\mathbf{G}}_{\mathbf{L}}(M) = 0$ for all proper $\mathbf{G}$-split Levi subgroups $\mathbf{L}$.

We denote the set of cuspidal $\mathbf{G}^{\mathrm{F}}$-modules by $\operatorname{Cusp}_A(\mathbf{G}^{\mathrm{F}})$ or just $\operatorname{Cusp}(\mathbf{G}^{\mathrm{F}})$ if the base ring is clear.

Definition 5.2.6. A *cuspidal pair* is a pair $(\mathbf{L},\Lambda)$ where $\mathbf{L}$ is an $\mathbf{G}$-split Levi subgroup and Λ is a cuspidal simple $A\mathbf{L}^F$-module. The *Harish-Chandra series* corresponding to such a pair defined by

$$ \operatorname{Irr}(\mathbf{G}^F|(\mathbf{L},\Lambda)) = \left\{ M \in \operatorname{Irr}\mathbf{G}^F \mid R^{\mathbf{G}}_{\mathbf{L}}(\Lambda) \twoheadrightarrow M \right\} / \sim . $$

Theorem 5.2.7. *(i) For any simple $A\mathbf{G}^{\mathrm{F}}$-*mod *M there exist a $\mathbf{G}$-split Levi subgroup $\mathbf{L}$ and a (simple) cuspidal $\mathbf{L}^F$-module Λ such that M is in the* head *of $R^{\mathbf{G}}_{\mathbf{L}}(\Lambda)$,* i.e. *such that there exists a $\mathbf{G}^F$-equivariant surjective map $R^{\mathbf{G}}_{\mathbf{L}}(\Lambda) \twoheadrightarrow M$.*

(ii) If $A = K$ is a field. Let $(\mathbf{L},\Lambda)$ and $(\mathbf{L}',\Lambda')$ be two cuspidal pairs. Then

$$ \operatorname{Irr}_K(\mathbf{G}^{\mathrm{F}}|(\mathbf{L},\Lambda)) \cap \operatorname{Irr}_K(\mathbf{G}^F|\left(\mathbf{L}',\Lambda'\right)) \neq \varnothing \Leftrightarrow \exists g \in \mathbf{G}^F \text{ such that } \left(\mathbf{L}',\Lambda'\right) = {}^g(\mathbf{L},\Lambda). $$

Proof. (i) Let $\mathbf{L}$ be a minimal Levi subgroup such that ${}^*R^{\mathbf{G}}_{\mathbf{L}}(M) \neq 0$, by transitivity of restriction, ${}^*R^{\mathbf{G}}_{\mathbf{L}}(M)$ is a cuspidal $A\mathbf{L}^{\mathrm{F}}$-module. Let Λ be any simple submodule of ${}^*R^{\mathbf{G}}_{\mathbf{L}}(M)$, we conclude from $\operatorname{Hom}_{A\mathbf{G}^{\mathrm{F}}}(R^{\mathbf{G}}_{\mathbf{L}}\Lambda, M) \cong \operatorname{Hom}_{A\mathbf{L}^{\mathrm{F}}}(\Lambda, {}^*R^{\mathbf{G}}_{\mathbf{L}}(M))$.

(ii) See [28, Theorem 5.3.7].

□

We conclude from the above theorem that when $A = K$ is a field, we have

Theorem 5.2.8. *The set of irreducible representations of $\mathbf{G}^{\mathrm{F}}$ is partitioned into Harish-Chandra series:*

$$ \operatorname{Irr}_K \mathbf{G}^F = \bigsqcup_{(\mathbf{L},\Lambda) \in \mathcal{C}(\mathbf{G}^{\mathrm{F}})/\mathbf{G}^F} \operatorname{Irr}_K \left(\mathbf{G}^F \mid (\mathbf{L},\Lambda)\right) $$

where $\mathcal{C}(\mathbf{G}^{\mathrm{F}})$ denote the set of cuspidal pairs of $\mathbf{G}^{\mathrm{F}}$.

We define the map $\operatorname{pr}_{\mathbf{G}^{\mathrm{F}}|(\mathbf{L},\Lambda)}$ as the projection map

$$ \operatorname{Irr}_K \mathbf{G}^F \to \operatorname{Irr}_K \left(\mathbf{G}^F \mid (\mathbf{L},\Lambda)\right). \tag{5.2.4} $$

5.3 Endomorphism algebras as finite Hecke algebras

From now on we work under the following assumption for the rest of this Chapter:

Assumption 5.3.1. $\mathbf{G}$ is a connected reductive group defined over $\mathbb{F}_q$, $A = K$ is a field of characteristic different from p and $\mathrm{End}_{K\mathbf{L}^{\mathrm{F}}}(\Lambda) = K$.

Now we define $N_{\mathbf{G}^{\mathrm{F}}}(\mathbf{L}, \Lambda) = \{n \in N_{\mathbf{G}^{\mathrm{F}}}(\mathbf{L}) \mid {}^n\Lambda \cong \Lambda\}$ and $W_{\mathbf{G}^{\mathrm{F}}}(\mathbf{L}, \Lambda) = N_{\mathbf{G}^{\mathrm{F}}}(\mathbf{L}, \Lambda)/\mathbf{L}^{\mathrm{F}}$. The endomorphism algebra associated with a cuspidal pair $(\mathbf{L}, \Lambda)$ is $\mathcal{H}(\mathbf{L}, \Lambda) = \mathrm{End}_{\mathbf{G}^{\mathrm{F}}}(R_{\mathbf{L}}^{\mathbf{G}}(\Lambda))$. From Mackey formula, we have

$$\mathcal{H}(\mathbf{L}, \Lambda) = \mathrm{End}_{\mathbf{G}^{\mathrm{F}}}(R_{\mathbf{L}}^{\mathbf{G}} N) \cong \bigoplus_{w \in W(\mathbf{L},\Lambda)^{\mathrm{F}}} \mathrm{Hom}(\Lambda, {}^{n_w}\Lambda)$$

where n_w is any representative of $w \in W(\mathbf{L}, \Lambda)^{\mathrm{F}}$ in $N(\mathbf{L}, \Lambda)^{\mathrm{F}}$. Each $\mathrm{Hom}(\Lambda, {}^{n_w}\Lambda)$ is one dimensional, denote the generator by γ_n. Define $e_{\mathbf{U}^{\mathrm{F}}}$ as $|\mathbf{U}^{\mathrm{F}}|^{-1} \sum_{u \in \mathbf{U}^{\mathrm{F}}} u$, and the linear maps

$$\begin{aligned} B_w : R_{\mathbf{L}}^{\mathbf{G}} &\to R_{\mathbf{L}}^{\mathbf{G}}\Lambda \\ g e_{\mathbf{U}^{\mathrm{F}}} \otimes_{K\mathbf{L}^{\mathrm{F}}} x &\mapsto g e_{\mathbf{U}^{\mathrm{F}}} n_w^{-1} e_{\mathbf{U}^{\mathrm{F}}} \otimes_{K\mathbf{L}^{\mathrm{F}}} \gamma_{n_w}(x). \end{aligned} \tag{5.3.1}$$

We know for $w \in W_{\mathbf{G}^{\mathrm{F}}}(\mathbf{L}, \Lambda)$, B_w form a basis of $\mathcal{H}(\mathbf{L}, \Lambda)$. From [28, Lemma 6.18], we have the following proposition

Proposition 5.3.2. *If $\ell(w) + \ell(w') = \ell(ww')$, then $B_w B_{w'} = \lambda(w, w') B_{ww'}$, where λ is a 2-cocycle given by $\gamma_{n_w} \circ \gamma_{n'_w} = \lambda(w, w')\gamma_{n_w n_{w'}}$.*

Define the Hom-functor as follows:

$$\begin{aligned} \mathcal{E}_G : K\mathbf{G}^{\mathrm{F}}\text{-}\mathrm{mod} &\to \mathcal{H}(\mathbf{L}, \Lambda)^{\mathrm{op}}\text{-}\mathrm{mod} \\ M &\mapsto \mathrm{Hom}_{\mathbf{G}^{\mathrm{F}}}(R_{\mathbf{L}}^{\mathbf{G}}(\Lambda), M). \end{aligned} \tag{5.3.2}$$

We will omit the sub-index of $\mathcal{E}_G$ if no confusion is caused. Following [29, Proposition 10.8], we have

Proposition 5.3.3. *The functor $\mathcal{E}$ induces a bijection*

$$\mathrm{Irr}_K(\mathbf{G}^{\mathrm{F}} | (\mathbf{L}, \Lambda)) \leftrightarrow \mathrm{Irr}_K(\mathcal{H}(\mathbf{L}, \Lambda)).$$

Theorem 5.3.4. *Let $\mathbf{L} \subset \mathbf{M} \subset \mathbf{G}$ be two split F-stable Levi components, $(\mathbf{L}, \Lambda)$ be a cuspidal pair in $\mathbf{G}$. Then the following two diagrams commute*

$$\begin{tikzcd}
\mathbb{Z}\operatorname{Irr}_K(\mathbf{G}^{\mathrm{F}}|(\mathbf{L},\Lambda)) \arrow[r, "\mathcal{E}_G"] \arrow[d, "\mathrm{pr}_{\mathbf{M}^{\mathrm{F}}|({}^w\mathbf{L},{}^w\Lambda)}\circ {}^*R^{\mathbf{G}}_{\mathbf{M}}"'] & \mathbb{Z}\operatorname{Irr}_K(\operatorname{End}_{\mathbf{G}^{\mathrm{F}}}(R^{\mathbf{G}}_{\mathbf{L}}\Lambda)) \arrow[d, "\operatorname{Res}^{\operatorname{End}_{\mathbf{G}^{\mathrm{F}}}(R^{\mathbf{G}}_{\mathbf{L}}\Lambda)}_{\operatorname{End}_{\mathbf{M}^{\mathrm{F}}}(R^{\mathbf{M}}_{{}^w\mathbf{L}}{}^w\Lambda)}"] \\
\mathbb{Z}\operatorname{Irr}_K(\mathbf{M}^{\mathrm{F}}|({}^w\mathbf{L},{}^w\Lambda)) \arrow[r, "\mathcal{E}_M"'] & \mathbb{Z}\operatorname{Irr}_K(\operatorname{End}_{\mathbf{M}^{\mathrm{F}}}(R^{\mathbf{M}}_{{}^w\mathbf{L}}{}^w\Lambda))
\end{tikzcd} \tag{5.3.3}$$

$$\begin{tikzcd}
\mathbb{Z}\operatorname{Irr}_K(\mathbf{G}^{\mathrm{F}}|(\mathbf{L},\Lambda)) \arrow[r, "\mathcal{E}_G"] & \mathbb{Z}\operatorname{Irr}_K(\operatorname{End}_{\mathbf{G}^{\mathrm{F}}}(R^{\mathbf{G}}_{\mathbf{L}}N)) \\
\mathbb{Z}\operatorname{Irr}_K(\mathbf{M}^{\mathrm{F}}|(\mathbf{L},\Lambda)) \arrow[r, "\mathcal{E}_M"'] \arrow[u, "R^{\mathbf{G}}_{\mathbf{M}}"] & \mathbb{Z}\operatorname{Irr}_K(\operatorname{End}_{\mathbf{M}^{\mathrm{F}}}(R^{\mathbf{M}}_{\mathbf{L}}\Lambda)) \arrow[u, "\operatorname{Ind}^{\operatorname{End}_{\mathbf{G}^{\mathrm{F}}}(R^{\mathbf{G}}_{\mathbf{L}}\Lambda)}_{\operatorname{End}_{\mathbf{M}^{\mathrm{F}}}(R^{\mathbf{M}}_{\mathbf{L}}\Lambda)}"']
\end{tikzcd} \tag{5.3.4}$$

where w belongs to $\mathbf{M}^{\mathrm{F}}\backslash\{w \in \mathcal{S}(\mathbf{M},\mathbf{L})^{\mathrm{F}} \mid w(\mathbf{L}^{\mathrm{F}}) \subset \mathbf{M}^{\mathrm{F}}\}/\mathbf{L}^{\mathrm{F}}$.

5.4 The Alvis-Curtis-Kawanaka duality

In this section we make the following assumption

Assumption 5.4.1. We require $A = K = \mathbb{C}$.

Let $K_0(\mathbb{C}\mathbf{G}^{\mathrm{F}}\text{-}\,\mathrm{mod})$ denote the Grothendieck group of the category of $\mathbb{C}\mathbf{G}^{\mathrm{F}}$-modules. We know this category is semisimple, a $\mathbb{Z}$-basis of this group is given by complex irreducible characters. Any element of $K_0(\mathbb{C}\mathbf{G}^{\mathrm{F}}\text{-}\,\mathrm{mod})$ can be described by a virtual character. In this section we will treat the characters and the $K_0(\mathbb{C}\mathbf{G}^{\mathrm{F}}\text{-}\,\mathrm{mod})$ in equal footing, the linear map induced by Harish-Chandra induction and restriction on them are denoted by the same notations $R^{\mathbf{G}}_{\mathbf{L}}$ and ${}^*R^{\mathbf{G}}_{\mathbf{L}}$.

Let $\mathbf{B}$ be a fixed F-stable Borel, and $\mathbf{T}$ is a fixed F-stable maximal torus of $\mathbf{B}$. Set $W = N_{\mathbf{G}}(\mathbf{T})^{\mathrm{F}}/\mathbf{T}^{\mathrm{F}}$, then $(\mathbf{B}^{\mathrm{F}}, N_{\mathbf{G}}(\mathbf{T})^{\mathrm{F}})$ is a BN-pair with Weyl group W. In the Coxeter system (W,S), denote by W_I the subgroup of W generated by I. Recall that the standard parabolic subgroup containing $\mathbf{B}^{\mathrm{F}}$ are $\mathbf{P}_I = \mathbf{B}^{\mathrm{F}} W_I \mathbf{B}^{\mathrm{F}}$.

Definition 5.4.2. The *Alvis-Curtis-Kawanaka duality* is the linear map on $K_0\left(\mathbb{C}\mathbf{G}^F\text{-}\,\mathrm{mod}\right)$ defined by

$$D_{\mathbf{G}} = \sum_{I\subset S,\ \mathrm{F}(I)=I} (-1)^{|I/\mathrm{F}|} R^{\mathbf{G}}_{\mathbf{L}_I} \circ {}^*R^{\mathbf{G}}_{\mathbf{L}_I}$$

where $\mathbf{L}_I$ is the standard F-stable Levi complement of F-stable standard parabolic subgroup $\mathbf{P}_I$.

The Borel subgroups, parabolic subgroups and Levi subgroups appearing in the following proposition are all assumed to be F-stable. Notations will have their usual meaning. We list some important properties of $D_{\mathbf{G}}$ following [28, Chapter 12]:

Proposition 5.4.3. *(i) (C. Curtis)* $D_{\mathbf{G}} \circ R_{\mathbf{L}}^{\mathbf{G}} = R_{\mathbf{L}}^{\mathbf{G}} \circ D_{\mathbf{L}}$ *and* ${}^*R_{\mathbf{L}}^{\mathbf{G}} \circ D_{\mathbf{G}} = D_{\mathbf{L}} \circ {}^*R_{\mathbf{L}}^{\mathbf{G}}$.

(ii) (C. Curtis) $D_{\mathbf{G}}$ *is self-adjoint (acting of characters).*

(iii) (C. Curtis) $D_{\mathbf{G}} \circ D_{\mathbf{G}}$ *is the identity map on* $K_0(\mathbb{C}\mathbf{G}^{\mathrm{F}}\text{-}\mathrm{mod})$.

(iv) Let $(\mathbf{L}, \Lambda)$ *be a cuspidal pair and let* γ *be the character of some irreducible representation such that* $\gamma \in \mathrm{Irr}(\mathbf{G}^{\mathrm{F}}|(\mathbf{L}, \Lambda))$. *Then* $(-1)^{r(\mathbf{L})} D_{\mathbf{G}}(\gamma) \in \mathrm{Irr}\left(\mathbf{G}^{F} \mid (\mathbf{L}, \Lambda)\right)$, *where* $r(\mathbf{L})$ *is the semisimple rank of* $\mathbf{L}$.

We see from (iv) of the above proposition that $D_{\mathbf{G}}$ preserves the $\mathrm{Irr}(\mathbf{G}^{\mathrm{F}}|(\mathbf{L}, \Lambda))$ up to ± 1.

5.5 A review of Howlett-Lehrer's duality formula

In this section we make the following assumption

Assumption 5.5.1. We require $A = K = \mathbb{C}$, $(\mathbf{G}, \mathrm{F})$ is *split*, *i.e.* F acts trivially on W.

In the case of $\mathbb{C}$-modules are considered, the quadratic relations for B_w are explicit and the 2-cocycle λ is trivial. There exists a set of involutions $S(\mathbf{L}, \Lambda) \subset W(\mathbf{L}, \Lambda)^{\mathrm{F}}$ (defined in section 5.3) such that $(S(\mathbf{L}, \Lambda), W(\mathbf{L}, \Lambda)^{\mathrm{F}})$ is a Coxeter system. Combining Lemma 4.2 and Theorem 5.9 of *loc.cit.*, we have the following theorem:

Theorem 5.5.2. *The endomorphism algebra* $\mathrm{End}_{\mathbf{G}^{\mathrm{F}}}(R_{\mathbf{L}}^{\mathbf{G}} N)$ *is a Hecke algebra, we have*

$$\mathcal{H}_{\mathbf{q}}(S(\mathbf{L}, \Lambda), W(\mathbf{L}, \Lambda)^{\mathrm{F}}) \otimes_f \mathbb{C} \cong \mathrm{End}_{\mathbf{G}^{\mathrm{F}}}(R_{\mathbf{L}}^{\mathbf{G}} N) \tag{5.5.1}$$

where f *denotes a specialization defined in [35, Lemma 4.2].*

Furthermore, in characteristic 0 case, by using Tits' deformation theorem we can sharpen Proposition 5.3.3 and Theorem 5.5.2:

Theorem 5.5.3 (Howlett-Lehrer)**.** *Let* $\mathbf{G}^{\mathrm{F}}$ *be a finite group of Lie type and* $(\mathbf{L}, \Lambda)$ *be a cuspidal pair, then*

$$\mathbb{C}W(\mathbf{L}, \Lambda)^{\mathrm{F}} \cong \mathcal{H}_{\mathbf{q}}(S(\mathbf{L}, \Lambda), W(\mathbf{L}, \Lambda)^{\mathrm{F}}) \otimes_{q_s \mapsto 1} \mathbb{C} \cong \mathcal{H}_{\mathbf{q}}(S(\mathbf{L}, \Lambda), W(\mathbf{L}, \Lambda)^{\mathrm{F}}) \otimes_f \mathbb{C} \cong \mathrm{End}_{\mathbf{G}^{\mathrm{F}}}(R_{\mathbf{L}}^{\mathbf{G}} N). \tag{5.5.2}$$

In particular, we have

$$\mathrm{Irr}_{\mathbb{C}}(W(\mathbf{L}, \Lambda)^{\mathrm{F}}) \leftrightarrow \mathrm{Irr}_{\mathbb{C}}(\mathbf{G}^{\mathrm{F}}|(\mathbf{L}, \Lambda)).$$

Moreover, in this setting the Alvis-Curtis-Kawanaka duality (Definition 5.4.2) becomes

$$D_{\mathbf{G}} = \sum_{I \subset S} (-1)^{|I|} R^{\mathbf{G}}_{\mathbf{L}_I} \circ {}^*R^{\mathbf{G}}_{\mathbf{L}_I}.$$

We now begin to introduce the setup in Howlett-Lehrer's series of works ([33], [34], [35]), where mainly complex representations are considered. The finite group $G = \mathbf{G}^{\mathrm{F}}$ ($\mathbf{G}(\mathbb{F}_q)$ in Howlett-Lehrer's work) has a (B, N)-pair with $B = P_0 = \mathbf{P}_0^{\mathrm{F}}$, $N = N_{\mathbf{G}}(\mathbf{T})^{\mathrm{F}}$ where $\mathbf{P}_0$ is a fixed Borel $\mathbb{F}_q$-subgroup and $\mathbf{T}$ is a fixed maximal $\mathbb{F}_q$-split torus of $\mathbf{P}_0$. The Borel subgroup determines a set of simple roots, a positive system $\Delta \subset \Sigma^+ \subset \Sigma = \Phi(\mathbf{T}^{\mathrm{F}}, \mathbf{G}^{\mathrm{F}})$ of W. The standard parabolic subgroups P_I $(= BW_IB)$of G are in bijection with subsets $I \subset \Delta$, and can be expressed as follows:

$$P_I = M_I U_I \quad \text{where} \begin{cases} M_I = \langle T, U_a \mid a \in \Sigma_I \rangle \\ U_I = \langle U_a \mid a \in \Sigma^+ \backslash \Sigma_I \rangle \end{cases}$$

where Σ_I is the sub-root system of Σ spanned by I.

From now on until the end of this Chapter, we fix a subset $I_0 \subset \Delta$, and set W_{I_0}, L_{I_0}, U_{I_0} and P_{I_0} as above. Let D be a representation of any Levi M_I, we will use the same notation D to denote the lift from M_I to P_I, for $w \in W$, ${}^wD(-)$ and ${}^wP_I$ are always understood as ${}^{n_w}D(-)$ and ${}^{n_w}P_I$ where n_w is any representative of w in N. Let $T_0 = T_{I_0}$ denote the maximal split torus contained in $Z(L_{I_0})$. We use Φ_{I_0} to denote the sub-root system spanned by I_0. The pair $(B \cap L_{I_0}, N \cap L_{I_0})$ provides BN-pair for L_{I_0}. We use Λ to denote an irreducible cuspidal representation of L_{I_0}, whose character is χ_Λ.

Proposition 5.5.4. *Denote by* $\mathrm{Aut}(T_0)$ *automorphism group of* T_0 *induced by conjugation of* G*, we have* $\mathrm{Aut}(T_0) \cong N_W(W_{I_0})/W_{I_0} \cong S_{I_0} := \{w \in W \mid wI_0 = I_0\}$.

Definition 5.5.5. The *ramification group* associated to cuspidal pair (L_{I_0}, Λ) is defined as follows

$$W(I_0, \Lambda) = \{w \in S_{I_0} \mid {}^w\Lambda = \Lambda\}.$$

When the set I_0 is clear, we may denote $W(I_0, \Lambda)$ shortly as $W(\Lambda)$.

Remark. $W(\Lambda)$ agrees with previously defined $W_{\mathbf{G}^{\mathrm{F}}}(\mathbf{L}, \Lambda)$ under the split Assumption 5.5.1.

Let $\hat{\Omega}(I_0) := \{a \in \Sigma \mid w(I_0 \cup \{a\}) \subset \Delta$ for some $w \in W\}$. For an element $a \in \hat{\Omega}(I_0)$, we define $v[a, I_0] = ut$ where u is the longest element of $W_{I_0 \cup \{a\}}$, and t is the longest element of W_{I_0}. We introduce the following definition from [35, Definition 3.6].

Definition 5.5.6 (Howlett-Lehrer)**.** Let $\Gamma(I_0, \Lambda)$ be the set of roots $a \in \Sigma \backslash I_0$ satisfying

(i) $w(I_0 \cup \{a\}) \subset \Delta$ for some $w \in W$.

(ii) $v[a, I_0] \in W(I_0, \Lambda)$.

(iii) With w as in (i), $I' = w(I_0 \cup \{a\})$, the induced representation $\mathrm{Ind}_{P_{wI_0} \cap M_{I'}}^{M_{I'}}({}^w\Lambda)$ has exactly two irreducible constituents ρ_1 and ρ_2 (we may assume $\dim(\rho_2) \geqslant \dim(\rho_1)$), and $p_a = \frac{\dim(\rho_2)}{\dim(\rho_1)} > 1$.

We denote by $\Gamma^+(I_0, \Lambda)$ the set $\Gamma(I_0, \Lambda) \cap \Sigma^+$, and denote by $R(I_0, \Lambda)$ the subgroup of $W(I_0, \Lambda)$ generated by $\{v[a, I_0] \mid a \in \Gamma(I_0, \Lambda)\}$. For any $w \in W$, set $N(w) := \{a \in \Sigma^+ \mid wa \in \Sigma^-\}$. We define $\Delta(I_0, \Lambda)$ as the set

$$\{a \in \Gamma^+(I_0, \Lambda) \mid N(v[a, I_0]) \cap \Gamma(I_0, \Lambda) = \{a\}\}.$$

We set $S(I_0, \Lambda) := \{v[a, I_0] \mid a \in \Delta(I_0, \Lambda)\}$. We may abbreviate $\Gamma(I_0, \Lambda)$ as $\Gamma(\Lambda)$ when no confusion is caused; we will use similar abbreviations for $R(I_0, \Lambda)$, $S(I_0, \Lambda)$ and $\Delta(I_0, \Lambda)$.

We have the following fact from [33, Lemma 2.7] and [35, Lemma 3.9]:

Lemma 5.5.7. *(i) The group $R(\Lambda)$ is a normal subgroup of $W(\Lambda)$ and acts as reflection group on $I_0^\perp$.*

(ii) The projection of $\Gamma(\Lambda)$ to $I_0^\perp$ is a root system for $R(\Lambda)$. The projection of $\Gamma^+(\Lambda)$ to $I_0^\perp$ is a positive system and the projection of $\Delta(\Lambda)$ is the corresponding fundamental system.

(iii) We can write $W(\Lambda)$ as $C(\Lambda) \ltimes R(\Lambda)$ where $C(\Lambda) = \{w \in W(\Lambda) \mid w\Gamma^+(\Lambda) = \Gamma^+(\Lambda)\}$.

Definition 5.5.8. (i) We define the length function $\ell_{I_0^\perp}$ for $w \in W(\Lambda)$ as the length function after projection to $\langle I_0 \rangle^\perp$, *i.e.* for $w = w_1 w_2$ with $w_1 \in C(\Lambda)$, $w_2 \in R(\Lambda)$, $\ell_{I_0^\perp}(w) = \ell_{R(\Lambda)}(w_2)$.

(ii) ([33, Definition 4.9]) For any $w \in W(\Lambda)$, we define

$$p_w = \prod_{a \in N(w) \cap \Gamma(\Lambda)} p_a.$$

We can now state the main theorem of [33, Theorem 4.14]

Theorem 5.5.9 (Howlett-Lehrer). *The algebra $E_G(\Lambda) = \mathrm{End}_G(\mathrm{Ind}_{P_{I_0}}^G(\Lambda))$ has a $\mathbb{C}$-basis (see [33, Section 4]) $\{T_w \mid w \in W(\Lambda)\}$ whose multiplication table satisfies (i) to (iv) below for all $w \in W(\Lambda), x \in C(\Lambda)$ and $a \in \Delta$. We set $v = v[a, I_0]$. Then we have*

(i)

$$T_w T_x = \mu(w, x) T_{wx},$$

(ii)

$$T_x T_w = \mu(x, w) T_{xw},$$

(iii)

$$T_v T_w = \begin{cases} T_{vw} & \left(\text{if } w^{-1}a \in \Gamma^+(\Lambda) \right), \\ p_a T_{vw} + (p_a - 1) T_w & \left(\text{if } w^{-1}a \notin \Gamma^+(\Lambda) \right), \end{cases}$$

(iv)

$$T_w T_v = \begin{cases} T_{wv} & (\text{ if } wa \in \Gamma^+(\Lambda))\,, \\ p_a T_{wv} + (p_a - 1)\, T_w & (\text{ if } wa \notin \Gamma^+(\Lambda))\,. \end{cases}$$

The 2-cocycle μ of $W(\Lambda)$ appearing in (i) and (ii) satisfies the properties that

(a) $\mu(xv, yw) = \mu(x, y)$ for all $x, y \in C(\Lambda), v, w \in R(\Lambda)$,

(b) the value $\mu(x, y)$ depends only on the cohomological class of μ,

(c) $\mu(x, 1) = \mu(1, x) = 1$ for all $x \in C(\Lambda)$.

We will work under the assumption that μ is trivial which is proved by Lusztig-Geck ([42] and [30]) for finite groups of Lie type. We have the following

Corollary 5.5.10. *The endomorphism algebra $E_G(\Lambda)$ has the basis consisting of $T_w = T_{r(w)}T_{c(w)}$ for all $w = c(w)r(w) \in W(\Lambda)$, where $c(w)$ (*resp. *$r(w)$) denotes the $C(\Lambda)$ (*resp. *$R(\Lambda)$) component in the decomposition $W(\Lambda) = C(\Lambda) \ltimes R(\Lambda)$.*

In the following sections, we will need study the structure of $\mathrm{End}_{L_I}(\mathrm{Ind}_{{}^wL_0}^{L_I}\, {}^w\Lambda)$ for different $I \subset \Delta$. Thus we need a description of the structure of the group $W(\Lambda) \cap W_I^w$ for $w \in V_\Lambda$ where $V_\Lambda := \{w \in W \mid wI_0 \subset \Delta,\ w\Gamma^+(\Lambda) \subset \Sigma^+\}$.

We state the following lemma ([35, Lemma 3.13]):

Lemma 5.5.11. *Let $w \in V_\Lambda$, $wI_0 \subset I \subset \Delta$, we denote by $R(\Lambda)_{w,I}$ be the parabolic subgroup of $R(\Lambda)$ generated by $S(\Lambda)_{w,I} := \{v[a, I_0] \mid a \in \Delta(\Lambda)_{w,I}\}$, where $\Delta(\Lambda)_{w,I} = \{a \in \Delta(\Lambda), wa \in \langle I\rangle\}$. Then we have the following semidirect product:*

$$W(\Lambda) \cap W_I^w = (W_I^w \cap C(\Lambda)) \ltimes R(\Lambda)_{w,I}.$$

For $w \in W$ such that $wI_0 \subset \Delta$, we define the intertwining operator $B_{\Lambda,w} : \mathrm{Ind}_{P_{I_0}}^G(\Lambda) \to \mathrm{Ind}_{{}^wP_{I_0}}^G({}^w\Lambda)$ by

$$(B_{\Lambda,w}f)(x) = \mathrm{Card}(U_{wI_0})^{-1} \sum_{y \in U_{wI_0}} f(n_w^{-1}yx).$$

From [33, Lemma 3.11 and Lemma 3.17], we know $B_{\Lambda,w}$ is a $\mathbb{C}G$-isomorphism. For $w \in V_\Lambda$, we define a map $\tau_w : E_G(\Lambda) \to E_G({}^w\Lambda)$ by $\tau_w(T) = B_{\Lambda,w}TB_{\Lambda,w}^{-1}$.

We can now state the following theorem from [35, Theorem 3.16] describing an important subalgebra of $E_G(\Lambda)$:

Theorem 5.5.12. *Let $w \in V_\Lambda, wI_0 \subset I \subset \Delta$. The image of $E_I({}^w\Lambda) := \mathrm{End}_{P_J}\left(\mathrm{Ind}_{{}^wP_{I_0}}^{P_I}({}^w\Lambda)\right)$ under the canonical injection $E_I({}^w\Lambda) \to E_G({}^w\Lambda)$ is the $\mathbb{C}$-linear span of $\{T'_v \mid v \in W_I \cap W({}^w\Lambda)\}$. Furthermore the $\mathbb{C}$-linear span of $\{T_v \mid v \in\ W_I^w \cap W(\Lambda)\}$ is a subalgebra of $E_G(\Lambda)$ which is mapped onto the image of $E_I({}^w\Lambda)$ by the isomorphism τ_w.*

The following theorem is a motivation for what follows:

Theorem 5.5.13 (L. Solomon)**.** *Let χ be a character of W. We have*

$$\sum_{I\subset S}(-1)^{|I|}\operatorname{Ind}_{W_I}^{W}\operatorname{Res}_{W_I}^{W}(\chi)=\widehat{\chi} \tag{5.5.3}$$

where $\widehat{\chi}$ is defined by $\chi(w)=\det(w)\cdot\chi(w)$.

Because $\det(w)=(-1)^{\ell(w)}$ we may see the right hand side of (5.5.3) as a character taking value $(-1)^{\ell(w)}\chi((w^{-1})^{-1})$ at w. Written in this way, and compared with $T_w^*=(-1)^{\ell(w)}q(w)T_{w^{-1}}^{-1}$, the $\widehat{}$ operation can be seen as the prototype of $*$ for characters of W.

In the article [34, Theorem and Corollary 1], R.B. Howlett and G.I. Lehrer proved a stronger version for the normalizer $N_W(W_{I_0})$ of a parabolic subgroup associated with a subset $I_0\subset S$ of W and its subgroups especially the ramification group $W(\Lambda)$.

Theorem 5.5.14 (R.B. Howlett, G.I. Lehrer)**.** *Let χ be a character of $W(\Lambda)$, we have the following equation of characters:*

$$\sum_{I\subset S}(-1)^{|I|}\sum_{w\in W_I\backslash C_{I_0}(I)/W(\Lambda)}\operatorname{Ind}_{W(\Lambda)\cap W_I^w}^{W(\Lambda)}(\operatorname{Res}_{W(\Lambda)\cap W_I^w}^{W(\Lambda)}(\chi))=\widehat{\chi}:=(-1)^{|I_0|}(-1)^{\ell_{I_0^{\perp}}(-)}\chi \tag{5.5.4}$$

where $I_0^{\perp}$ is its orthogonal complement of I_0. For any $I\subset S$, $C_{I_0}(I)$ is defined as the set $\{w\in W\mid wI_0\subset\langle I\rangle\}$.

We include a detailed proof of this Theorem in Section A.3 in the Appendix A.

Remark. We use the same notation as [35], which is different from [34] up to an inverse.

If we take $I_0=\varnothing$ and replace $W(\Lambda)$ by W, we see that (5.5.3) is a special case of (5.5.4).

5.6 Restriction of the ACK duality to a Harish-Chandra series

We work under the same assumption as previous section. Let us fix a cuspidal pair (L_{I_0},Λ) with L_{I_0} a fixed standard Levi subgroup, and choose $\pi\in\operatorname{Irr}_{\mathbb{C}}(G^{\mathrm{F}}|(L_{I_0},\Lambda))$. We abbreviate L_{I_0} by L_0. We study the restriction of Alvis-Curtis-Kawanaka (ACK) duality $\mathrm{D}_G=\sum_{I\subset S}(-1)^{|I|}R_{L_I}^G\circ{}^*R_{L_I}^G$ to $\operatorname{Irr}_{\mathbb{C}}(G^{\mathrm{F}}|(L_0,\Lambda))$ by let it act on π. We know π is a simple quotient of $R_{L_0}^G(\Lambda)$: $R_{L_0}^G(\Lambda)\twoheadrightarrow\pi$. Now applying first ${}^*R_{L_I}^G$, by Corollary 5.2.4 we have ${}^*R_{L_I}^G\circ R_{L_0}^G(\Lambda)$ equals a sum of the form $R_{L_I\cap{}^wL_0}^{L_I}\circ{}^*R_{L_I\cap{}^wL_0}^{{}^wL_0}\circ\operatorname{ad}(w)(\Lambda)$ for $w\in W_I\backslash W/W_{I_0}$. Then ${}^*R_{L_I}^G(\pi)$ is quotient of $\sum_{w\in\mathfrak{S}_I}R_{{}^wL_0}^{L_I}({}^w\Lambda)$, where $\mathfrak{S}_I:=\{w\in W_I\backslash W/W_{I_0}\mid w(I_0)\subset I\}$. We conclude that the images of π under ${}^*R_{L_I}^G$ are contained in the subsets $\operatorname{Irr}_{\mathbb{C}}(L_I|({}^wL_0,{}^w\Lambda))$ for $w\in\mathfrak{S}_I$. We now adapt the diagrams (5.3.3) and (5.3.4) accordingly for each w in this set:

$$\begin{tikzcd}
\mathbb{Z}\operatorname{Irr}_{\mathbb{C}}(G|(L_0,\Lambda)) \arrow[r, "\mathcal{E}_G"] \arrow[d, "\mathrm{pr}_{L_I|({}^wL_0,{}^w\Lambda)}\circ{}^*R^G_{L_I}"'] & \mathbb{Z}\operatorname{Irr}_{\mathbb{C}}(\operatorname{End}_G(R^G_{L_0}\Lambda)) \arrow[d, "\operatorname{Res}^{\operatorname{End}_G(R^G_{L_0}\Lambda)}_{\operatorname{End}_{L_I}(R^{L_I}_{{}^wL_0}{}^w\Lambda)}"] \\
\mathbb{Z}\operatorname{Irr}_{\mathbb{C}}(L_I|({}^wL_0,{}^w\Lambda)) \arrow[r, "\mathcal{E}_M"'] & \mathbb{Z}\operatorname{Irr}_{\mathbb{C}}(\operatorname{End}_{L_I}(R^{L_I}_{{}^wL_0}{}^w\Lambda))
\end{tikzcd} \tag{5.6.1}$$

$$\begin{tikzcd}
\mathbb{Z}\operatorname{Irr}_{\mathbb{C}}(G|(L_0,\Lambda)) \arrow[r, "\mathcal{E}_G"] & \mathbb{Z}\operatorname{Irr}_{\mathbb{C}}(\operatorname{End}_G(R^G_{L_0}\Lambda)) \\
\mathbb{Z}\operatorname{Irr}_{\mathbb{C}}(L_I|({}^wL_0,{}^w\Lambda)) \arrow[r, "\mathcal{E}_M"'] \arrow[u, "R^G_{L_I}"] & \mathbb{Z}\operatorname{Irr}_{\mathbb{C}}(\operatorname{End}_{L_I}(R^{L_I}_{{}^wL_0}{}^w\Lambda)) \arrow[u, "\operatorname{Ind}^{\operatorname{End}_G(R^G_{L_0}\Lambda)}_{\operatorname{End}_{L_I}(R^{L_I}_{{}^wL_0}{}^w\Lambda)}"']
\end{tikzcd} \tag{5.6.2}$$

where $\mathrm{pr}_{G|(L,N)}$ denote the projection functor from $\operatorname{Irr}_{\mathbb{C}}(G)$ to the Harish-Chandra series $\operatorname{Irr}_{\mathbb{C}}(G|(L,N))$, the explicit formula is given in [5] and [6]. We use Theorem 5.2.7 (ii) to identify $\operatorname{Irr}_{\mathbb{C}}(G|({}^wL_0,{}^w\Lambda))$ with $\operatorname{Irr}_{\mathbb{C}}(G|(L_0,\Lambda))$. We have shown in the above discussion that the map ${}^*R^G_{L_I}$ restricts to $\operatorname{Irr}_{\mathbb{C}}(G|(L_0,\Lambda))$ can be written as sum

$$\sum_{w\in\mathfrak{S}_I}\mathrm{pr}_{L_I|({}^wL_0,{}^w\Lambda)}\circ{}^*R^G_{L_I}.$$

The duality functor has the form

$$\mathrm{D}_G=\sum_{I\subset S}(-1)^{|I|}\Big(\sum_{w\in\mathfrak{S}_I}R^G_{L_I}\circ\mathrm{pr}_{L_I|({}^wL_0,{}^w\Lambda)}\circ{}^*R^G_{L_I}\Big). \tag{5.6.3}$$

Comparing with the formula in Theorem 5.5.14 and using the bijection

$$\operatorname{Irr}_{\mathbb{C}}(W(\Lambda))\leftrightarrow\operatorname{Irr}_{\mathbb{C}}(G|(L_0,\Lambda)),$$

we see that (5.6.3) is the counterpart of left hand side of (5.5.4) in Theorem 5.5.14 on the finite group side. By the two commutative diagrams (5.3.3) and (5.3.4), we find on the Hecke algebra side, the involution is

$$\mathrm{D}_{\mathcal{H}}=\sum_{I\subset S}(-1)^{|I|}\Big(\sum_{w\in\mathfrak{S}_I}\operatorname{Ind}^{\operatorname{End}_G(R^G_{L_0}\Lambda)}_{\operatorname{End}_{L_I}(R^{L_I}_{{}^wL_0}{}^w\Lambda)}\circ\operatorname{Res}^{\operatorname{End}_G(R^G_{L_0}\Lambda)}_{\operatorname{End}_{L_I}(R^{L_I}_{{}^wL_0}{}^w\Lambda)}\Big). \tag{5.6.4}$$

We will study the explicit expression for it in the next section.

5.7 The involution for the endomorphism algebras

The following is an analogue of Theorem 5.5.14 for endomorphism algebras of induced modules whose proof is inspired by Theorem 4.0.2 and uses properties from articles [33], [34] and [35] by R.B. Howlett, G.I. Lehrer. Recall from 5.5.7 that the ramification group $W(\Lambda)$ has a decomposition

$$W(\Lambda) = C(\Lambda) \ltimes R(\Lambda),$$

where

$$C(\Lambda) = \{w \in W(\Lambda) \mid w\Gamma^+(\Lambda) = \Gamma^+(\Lambda)\}.$$

Assumption 5.7.1. The group $C(\Lambda)$ is trivial.

This assumption is satisfied, for example, in [33, Example 4.15]. We have our second main Theorem as follows:

Theorem 5.7.2. *We work under the Assumption 5.7.1. Let M^* denote the module M endowed with the twisted action of $E_G(\Lambda)$ defined by*

$$T_w^* = (-1)^{|I_0|+\ell_{I_0^\perp}(w)} p_w T_{w^{-1}}^{-1},$$

where the p_w is defined in Definition 5.5.8. Then we have the following equality in the Grothendieck group of $E_G(\Lambda)$-modules:

$$\sum_{I \subset S} (-1)^{|I|} \sum_{w \in W_I \backslash C_{I_0}(I)/W(\Lambda)} [\mathrm{Ind}_{E_I'}^{E_G(\Lambda)} [\mathrm{Res}_{E_I'}^{E_G(\Lambda)}(M)]] = [M^*], \tag{5.7.1}$$

where $E_G(\Lambda)$ is defined as $\mathrm{End}_G(\mathrm{Ind}_{P_{I_0}}^G(\Lambda))$ with Λ viewed as a P_{I_0}-module through the natural lifting $P_{I_0} \to L_{I_0}$ and E_I' is the subalgebra of $E_G(\Lambda)$ spanned by $\{T_v \mid v \in W(\Lambda) \cap W_I^w\}$ (See Theorem 5.5.12).

If we let $I_0 = \varnothing$, then $C_{I_0}(I) = W$. In this case, $P_{I_0} = P_\varnothing = P_0$ is the minimal parabolic subgroup and $L_{I_0} = T$. We take Λ to be trivial character of T then $W(1) = W$, and we have $E_G(1) = \mathrm{End}_G(\mathrm{Ind}_{P_0}^G(1))$ is just the finite Hecke algebra H associated with (W, S), and E_I' becomes the subalgebra of H spanned by $\{T_v \mid v \in W_I\}(= H_I)$. We see that (5.7.1) degenerates to (4.0.2) for finite Hecke algebra.

Let us denote V the real vector space spanned by simple roots S. By Bourbaki [17, Chapter IV] (or see Theorem A.2.5 in Appendix A) we know that the Coxeter complex of W is the simplicial subdivision of the unit sphere $S(V)$. The simplexes therefore correspond to W-translations of the fundamental chamber, *i.e.* they are the collection $\{wC_I \mid w \in W\}$, where

$$C_I = \{v \in V \mid (v, \alpha) > 0 \text{ for } \alpha \in S - I,\ (v, \alpha) = 0 \text{ for } \alpha \in I\}.$$

There are a lemma and a corollary from [34] which imply the geometry we need to deal with. For a more detailed treatment, please see the Appendix A.

Let $C_{I_0}(I)$ be defined as the set $\{w \in W \mid wI_0 \subset \langle I\rangle\}$.

Lemma 5.7.3 (Howlett-Lehrer). *The set $R_{I_0} = \{wC_I \mid I \subset S, w \in C_{I_0}(I)\}$ is precisely the set of those W-regions contained in $I_0^{\perp}$.*

Corollary 5.7.4. *The poset $\Gamma_{I_0} = \{wW_I \mid I \subset S, w \in C_{I_0}(I)\}$ defines a subcomplex of the Coxeter complex Γ of W, which is a simplicial subdivision of $S(I_0^{\perp})$.*

Further, we know that $C_{I_0}(I)$ can be seen as a union of $(N_W(W_I), N_W(W_{I_0}))$ double cosets, hence a union of $(W_I, W(\Lambda))$ double cosets. From [35, Corollary 5.5], we know each $(W_I, W(\Lambda))$ double coset contains an element w with $wI_0 \subset I$ and $w \in V_\Lambda$. We will assume the choice of such w in the proof.

We see that the left hand side of (5.7.1) is very close to (but still different from) the left hand side of (5.6.4). Namely, $\mathfrak{S}_I$ of (5.6.4) by definition equals $W_I\backslash C_{I_0}(I)/W_{I_0}$; the big endomorphism algebra $\mathrm{End}_G(R^G_{L_0}\Lambda)$ in (5.6.4) equals $E_G(\Lambda)$ in in (5.6.4), and $\mathrm{End}_{L_I}(R^{L_I}_{{}^wL_0}{}^w\Lambda)$ (5.6.4) differs slightly from $\mathrm{End}_{P_J}\left(\mathrm{Ind}^{P_I}_{{}^wP_{I_0}}({}^w\Lambda)\right)$ which is isomorphic to E'_I in (5.7.1).

We can now prove the Theorem 5.7.2.

Proof. We shall imitate the proof by S-I. Kato to prove this theorem. We work under the assumption that $C(\Lambda)$ is trivial. Under this assumption, we have $W(\Lambda) = R(\Lambda)$, thus the endomorphism algebra $E_G(\Lambda)$ is isomorphic to $H(R(\Lambda), S(\Lambda))$ where

$$S(\Lambda) = \{v[a, I_0] \mid a \in \Gamma^+(I_0, \Lambda),\ N(v[a, I_0]) \cap \Gamma(I_0, \Lambda) = \{a\}\}.$$

Moreover, we have $W(\Lambda) \cap W_I^w = R(\Lambda)_{w,I}$ (see Lemma 5.5.11) which is a standard parabolic subgroup of $R(\Lambda)$. By definition, we have

$$\begin{aligned}
&\sum_{w\in W_I\backslash C_{I_0}(I)/W(\Lambda)} \mathrm{Ind}^{E_G(\Lambda)}_{E'_I}(\mathrm{Res}^{E_G(\Lambda)}_{E'_I}(M))\\
&= \sum_{w\in W_I\backslash C_{I_0}(I)/W(\Lambda)} E_G(\Lambda)\otimes_{E'_I} M\\
&= \begin{cases} 0, \quad (\text{in this case } C_{I_0}(I) = \varnothing) & \text{if } |I| < |I_0|,\\ \displaystyle\sum_{w\in W_I\backslash C_{I_0}(I)/W(\Lambda)} E_G(\Lambda)\otimes_K M/\sum_{s\in S(\Lambda)_{w,I}} \langle hT_u\otimes \pi(T_u)^{-1}m - h\otimes m\rangle, & \text{if } |I| \geqslant |I_0|.\end{cases}
\end{aligned} \tag{5.7.2}$$

To simplify notation, we will denote $L_s = hT_s \otimes \pi(T_s)^{-1}m - h\otimes m$.

Let us define

$${}_w\pi_I^{I'} : E_G(\Lambda)\otimes_K M/\sum_{s\in S(\Lambda)_{w,I}} L_s \to E_G(\Lambda)\otimes_K M/\sum_{s\in S(\Lambda)_{w,I'}} L_s$$

as the natural projection where I' is obtained by adding one more element wb from the subset $w(\Delta(\Lambda)\backslash\Delta(\Lambda)_{w,I}) \subset \Delta$. Here ${}_w\pi_I^{I'}$ is an $E_G(\Lambda)$-homomorphism. We also define

$$\epsilon_I^{I'} : \bigwedge^{|I|-|I_0|}(K^{I\backslash wI_0}) \to \bigwedge^{|I'|-|I_0|}(K^{I'\backslash wI_0})$$

as the natural isomorphism given by $v \to v \wedge c$, $c \in I'\backslash I$. Here K^X is the free K-modules for any set X, and put $\bigwedge^{|I|}(K^I) = K$ for $I = \varnothing$.

We consider the following complex:

$$0 \to C_0 \xrightarrow{d_0} C_1 \xrightarrow{d_1} \dots \xrightarrow{d_{i-1}} C_i \xrightarrow{d_i} \dots \xrightarrow{d_{|S|-1}} C_{|S|} \to 0 \tag{5.7.3}$$

where

$$C_i = \bigoplus_{\substack{|I|=i,\\ w\in W_I\backslash C_{I_0}(I)/W(\Lambda)}} (E_G(\Lambda) \otimes M / \sum_{s\in S(\Lambda)_{w,I}} L_s) \otimes \bigwedge^{i-|I_0|}(K^{I\backslash wI_0}),$$

and $d_i = \bigoplus_{\substack{|I|=i,\\ w\in W_I\backslash C_{I_0}(I)/W(\Lambda)}} {}_w\pi_I^{I'} \otimes \epsilon_I^{I'}$ for $i = |I| \geqslant |I_0|$; and $C_i = 0$, $d_i = 0$ otherwise.

Let us identify $M^{W(\Lambda)} = K^{W(\Lambda)} \otimes_K M$ with $E_G(\Lambda) \otimes_K M$ by the linear map:

$$\varphi : M^{W(\Lambda)} \to E_G(\Lambda) \otimes_K M \quad \text{defined by } \varphi((m_w)_{w\in W(\Lambda)}) = \sum_{w\in W(\Lambda)} T_w \otimes \pi(T_w)^{-1} m_w, \tag{5.7.4}$$

and forget about the $E_G(\Lambda)-$module structure on $K^{W(\Lambda)} \otimes M$.

We use the same argument as Lemma 4.1.1 to obtain for $s \in S(\Lambda)_{w,I}$,

$$K_s^{W(\Lambda)} = \{(x_w)_{w\in W(\Lambda)} \in K^{W(\Lambda)} \mid x_{ws} = -x_w,\ w \in W(\Lambda)\},$$

we have $\varphi^{-1}(L_s) = K_s^{W(\Lambda)} \otimes_K M$. Now the following holds:

$$(K^{W(\Lambda)}/\sum_{s\in S(\Lambda)_{w,I}} K_s^{W(\Lambda)}) \otimes M \cong E_G(\Lambda) \otimes_K M/\sum_{s\in S(\Lambda)_{w,I}} L_s.$$

Using the results we get so far, we can write the complex as

$$
\begin{aligned}
&0 \to \dots 0 \xrightarrow{d_{|I_0|-1}} \bigoplus_{\substack{|I|=|I_0|,\\ w\in W_I\backslash C_{I_0}(I)/W(\Lambda)}} (K^{W(\Lambda)}/\sum_{s\in S(\Lambda)_{w,I}} K_s^{W(\Lambda)}) \otimes \bigwedge^{|I|-|I_0|}(K^{I\backslash wI_0}) \xrightarrow{d_{|I_0|}} \\
&\xrightarrow{d_{|I_0|}} \bigoplus_{\substack{|I|=|I_0|+1,\\ w\in W_I\backslash C_{I_0}(I)/W(\Lambda)}} (K^{W(\Lambda)}/\sum_{s\in S(\Lambda)_{w,I}} K_s^{W(\Lambda)}) \otimes \bigwedge^{|I|-|I_0|}(K^{I\backslash wI_0}) \xrightarrow{d_{|I_0|+1}} \dots \\
&\dots \xrightarrow{d_{|S|-1}} C_{|S|} \to 0
\end{aligned}
\tag{5.7.5}
$$

where in the second line of (5.7.5), the s is the unique element of $S(\Lambda)_{w,I}$.

We can further identify $K^{W(\Lambda)}/\sum_{s\in S(\Lambda)_{w,I}} K_s^{W(\Lambda)}$ with $K[W(\Lambda)/W_{\Delta(\Lambda)_{w,I}}]$ by the map

$$(x_w)_{w\in W(\Lambda)} \mapsto \sum_{w\in W(\Lambda)} x_w \cdot wW_{\Delta(\Lambda)_{w,I}}.$$

Then uses the complex by [58] for simplicial decomposition of $(|S|-|I_0|-1)$-dimensional sphere, or see [27, the main theorem], we have

$$D[M] = \sum_i (-1)^i [C_i] = (-1)^{|I_0|}[\operatorname{Ker} d_{|I_0|}] = (-1)^{|I_0|}[\bigcap_{s\in S(\Lambda)} L_s],$$

By Lemma 4.1.1 and the same argument as (4.1.8), we know that

$$\bigcap_{s\in S(\Lambda)} L_s = (\mathrm{id}\otimes\pi)(\chi)(1\otimes M),$$

where

$$\chi_{I_0^\perp} = \sum_{w\in W(\Lambda)} (-1)^{\ell_{I_0^\perp}(w)} T_w \otimes T_w^{-1}.$$

We now need to show that $(-1)^{|I_0|}\chi_{I_0^\perp}(1\otimes M)$ is isomorphic to M^*, or equivalently:

$$(T_w\otimes 1)\chi_{I_0^\perp} = \chi_{I_0^\perp}(1\otimes T_w^*) = \chi_{I_0^\perp}(1\otimes (-1)^{\ell_{I_0^\perp}(w)} p_w T_{w^{-1}}^{-1}).$$

Hence we need to show χ intertwines $E_G(\Lambda)$-action and twisted $E_G(\Lambda)$-action as above, but this is done using Lemma 4.2.1 for the finite case.

∎

Chapter 6

Representation theory of p-adic groups via Hecke algebras

6.1 Notation and preliminaries

In this chapter, we start by introduce notations for this section. Let $\mathbb{G}$ be a split group over a non archimedean local field F, $G = \mathbb{G}(F)$ denote its F-rational points.

Let $H \subset G$ be a subgroup and (ρ, V_ρ) a smooth representation of H (*i.e.* for any $v \in V$, there is a compact open subgroup K fixing v), we write $(\mathrm{ind}_H^G(\rho), \mathrm{ind}_H^G(V_\rho))$ for the compactly induced representation. For $g \in G$, ${}^g(\text{-})$ denote the action $g(\text{-})g^{-1}$ on G, ${}^g\rho$ denote the representation $\rho({}^{g^{-1}}(\text{-}))$. From now on until the end of this paragraph, we assume H is open in G. For a smooth (π, V) representation of G, we write $(\mathrm{res}_H^G(\pi), \mathrm{res}_H^G(V))$ for the restriction functor. We have an adjoint pair $(\mathrm{ind}_H^G, \mathrm{res}_H^G)$ in the sense that there exists a natural equivalence $\mathrm{Hom}_G(\mathrm{ind}_H^G(\rho), -) \cong \mathrm{Hom}_H(\rho, \mathrm{res}_H^G(-))$.

For any smooth representation (ρ, V_ρ) let $(\rho^\vee, V_\rho^\vee)$ denote the smooth contragredient of (ρ, V_ρ).

Let us fix a minimal F-subgroup P_0 of G and a maximal F-split torus T contained in P_0. A parabolic subgroup P of G, with Levi subgroup L is said to be *standard* if $P \supset P_0$ and $L \supset T$. We have explicitly $P = LU$ where U is the unipotent radical and we also write the opposite Levi by $\overline{P} = L\overline{U}$. Let $\mathcal{P}(L)$ denote the set of parabolic subgroups with Levi component L. Let $W(G,T) := N_G(T)/T$ denote the (finite) Weyl group of G, R the set of roots and S the set of simple roots determined by the choice of P_0. For $I \subset S$, let P_I denote the standard parabolic F-subgroup of G determined by I, and L_I the Levi subgroup of P_I. We write $\widehat{T}$ for the complex torus dual to T, $\widehat{G}$ for the Langlands dual group of G, W_F for the Weil group of F, I_F for the inertial group of F.

For a smooth (ρ, V_ρ) representation of a Levi subgroup $L \subset P = LU$, ρ extends to a representation of P by letting U act trivially. Let $(\mathrm{Ind}_P^G(\rho), \mathrm{Ind}_P^G(V_\rho))$ denote the unnormalized induction. For a smooth representation (π, V) of G, let (π_U, V_U) denote the un-normalized Jaquet module of (π, V) and $j_U : V \to V_U$ denote the quotient map. We denote the normalized induction and normalized restriction(Jacquet module) of (ρ, V_ρ) by $(\mathrm{i}_P^G(\rho), \mathrm{i}_P^G(V_\rho))$ and $(\mathrm{r}_P^G(\rho), \mathrm{r}_P^G(V_\rho))$ (or $(\mathrm{r}_U(\rho), \mathrm{r}_U(V_\rho))$) respectively. We have two adjoint pairs: (r_U, i_P^G), where there exists a natural equivalence $\mathrm{Hom}_G(-, \mathrm{i}_P^G(\rho)) \cong \mathrm{Hom}_L(r_U(-), \rho)$ and $(\mathrm{i}_P^G, r_{\overline{U}})$ with $\mathrm{Hom}_G(\mathrm{i}_P^G(\rho), -) \cong \mathrm{Hom}_L(\rho, r_{\overline{U}}(-))$ (the second adjoint theorem, see [51, VI.9]).

Let M be a Levi subgroup of G, $\mathrm{Rep}(M)$ denote the category of all smooth complex representations of M, $\mathrm{Rep}_{\mathrm{f}}(M)$ denote the subcategory of finite length representations, and $\mathrm{Irr}(M)$ denote the set of isomorphism classes of smooth irreducible representations. Let $\mathfrak{R}(G)$ be the Grothendieck group of $\mathrm{Rep}(G)$ and for $\pi \in \mathrm{Rep}(G)$, denote by $[\pi]$ its image in $\mathfrak{R}(G)$.

6.2 The Aubert-Zelevinsky duality

For $\pi \in \mathrm{Rep}_{\mathrm{f}}(G)$ we follow [7], consider the map

$$\begin{aligned} \mathrm{D}_G : \mathfrak{R}(G) &\to \mathfrak{R}(G) \\ [\pi] &\mapsto [\sum_{I \subset S} (-1)^{|I|}\, \mathrm{i}_{P_I}^G \circ \mathrm{r}_{P_I}^G(\pi)]. \end{aligned} \tag{6.2.1}$$

It satisfies the following basic properties which are proved by [7, Theorem 1.7]:

(1) We have $\mathrm{D}_G((-)^\vee) = (\mathrm{D}_G(-))^\vee$.

(2) For any subset $J \subset S$, $\mathrm{D}_G \circ \mathrm{i}_{P_J}^G = \mathrm{i}_{P_J}^G \circ \mathrm{D}_{L_J}$.

(3) $\mathrm{D}_G^2 = \mathrm{Id}$.

(4) If π is irreducible cuspidal, then $\mathrm{D}_G(\pi) = (-1)^{|S|}\pi$.

6.3 The Bernstein decomposition

Let $\pi \in \mathrm{Irr}(G)$, there exists a parabolic subgroup $P = LU$ of G and a supercuspidal irreducible representation σ of L such that π embeds in $\mathrm{i}_P^G \sigma$. The pair (L, σ) is unique up to G-conjugacy. The G-conjugacy class $(L, \sigma)_G$ of (L, σ) is called the *supercuspidal support* of π. We denote by Sc the map defined by

$$\begin{aligned} \mathrm{Irr}(G) &\to \{ \text{ equivalence classes of supercuspidal pairs}\} \\ \pi &\mapsto (L, \sigma) \text{ if } \pi \cong \text{ a } G\text{-subquotient of } \mathrm{i}_P^G(\sigma). \end{aligned} \tag{6.3.1}$$

Let L be a Levi subgroup of a parabolic subgroup P of G, and let $\mathfrak{X}_{\mathrm{nr}}(L)$ denote the group of unramified characters of L. Given two cuspidal pairs $(L_i, \sigma_i)\,(i = 1, 2)$, they are *inertially equivalent* if there exists a $g \in G$ and $\nu \in \mathfrak{X}_{\mathrm{nr}}(L_2)$ such that:

$$L_2 = {}^gL_1 \quad \text{and} \quad {}^g\sigma_1 \simeq \sigma_2 \otimes \nu.$$

We write $[L,\sigma]_G$ for the inertial equivalence class of (L,σ) (sometimes omitting the sub-index if no confusion is caused) and $\mathfrak{B}(G)$ for the set of all inertial equivalence classes. If M is a proper Levi subgroup of G, L is a Levi subgroup of M and σ is a supercuspidal representation of L, then such $\mathfrak{s}_M = [L,\sigma]_M \in \mathfrak{B}(M)$ determines a $\mathfrak{s}_G = [L,\sigma]_G \in \mathfrak{B}(G)$ naturally. If (π, V) is an irreducible smooth representation of G, one defines the *inertial support* $\mathfrak{I}(\pi)$ of (π, V) to be the inertial equivalence class of the supercuspidal support of π.

Now fix a class $\mathfrak{s} \in \mathfrak{B}(G)$. We denote by $\mathrm{Rep}^{\mathfrak{s}}(G)$ the full subcategory of $\mathrm{Rep}(G)$ defined as follows:

Let $(\pi, \mathcal{V}) \in \mathrm{Rep}(G)$. Then $(\pi, \mathcal{V}) \in \mathrm{Rep}^{\mathfrak{s}}(G)$ if and only if every irreducible G-subquotient π_0 of π satisfies $\mathfrak{I}(\pi_0) = \mathfrak{s}$.

The subcategories $\mathrm{Rep}^{\mathfrak{s}}(G)$, $\mathfrak{s} \in \mathfrak{B}(G)$ split the category $\mathrm{Rep}(G)$. This is the Bernstein decomposition of $\mathrm{Rep}(G)$ (see [13]):

$$\mathrm{Rep}(G) = \prod_{\mathfrak{s}\in\mathfrak{B}(G)} \mathrm{Rep}^{\mathfrak{s}}(G) \tag{6.3.2}$$

of the subcategories $\mathrm{Rep}^{\mathfrak{s}}(G)$ as $\mathfrak{s}$ ranges through $\mathfrak{B}(G)$. Let

$$\mathrm{pr}_G^{\mathfrak{s}} : \mathrm{Rep}(G) \to \mathrm{Rep}^{\mathfrak{s}}(G) \tag{6.3.3}$$

denote the projection functor.

For $\mathfrak{s} = [L,\sigma]_G \in \mathfrak{B}(G)$, following [9, section 3.2] we define $\mathrm{N}_G(\mathfrak{s}_L) := \{g \in G : {}^gL = L$ and ${}^g\sigma \simeq \chi \otimes \sigma$, for some $\chi \in \mathfrak{X}_{\mathrm{nr}}(L)\}$, where $\mathfrak{s}_L = [L,\sigma]_L \in \mathfrak{B}(L)$ and denote by $W_G^{\mathfrak{s}}$ the extended finite Weyl group $\mathrm{N}_G(\mathfrak{s}_L)/L$ and $W_{\mathrm{aff}}^{\mathfrak{s}}$ the corresponding affine version.

6.4 Hecke algebras associated with Bernstein blocks

Let K be a compact open subgroup of G and (ρ, V_ρ) a representation of K, now recall the definition of Hecke algebra associated with (K,ρ).

Definition 6.4.1. Let $\mathcal{H}(G,\rho)$ denote the space of compactly supported functions

$$\phi : G \to \mathrm{End}_{\mathbb{C}}(V_\rho)$$

satisfying

$$\phi(k_1 g k_2) = \rho(k_1) \circ \phi(g) \circ \rho(k_2)$$

for all k_1, $k_2 \in K$ and $g \in G$. For ϕ_i, $\phi_2 \in \mathcal{H}(G,\rho)$, the standard convolution product

$$(\phi_1 * \phi_2)(x) := \int_G \phi_1(y) \circ \phi_2(y^{-1}x)dy$$

gives $\mathcal{H}(G,\rho)$ a structure of $\mathbb{C}$-algebra.

There exists an anti-isomorphism from $\mathcal{H}(G,\rho)$ to $\mathcal{H}(G,\rho^\vee)$ by sending $\phi(\text{-})$ to $\phi((\text{-})^{-1})^\vee$. Our definition for $\mathcal{H}(G,\rho)$ differs from [52, section 7] and [20, 2.4] by a contragredient, following *loc.cit* 2.6 we have an algebra isomorphism

$$\mathcal{H}(G,\rho) \cong \mathrm{End}_G(\mathrm{ind}_K^G(\rho)). \tag{6.4.1}$$

where $\mathrm{End}_G(\mathrm{ind}_K^G(\rho))$ denotes the right G-endomorphism.

6.5 An equivalence of categories

Let L be a fixed Levi subgroup of G, σ be a fixed supercuspidal representation of L and $\mathfrak{s} = [L,\sigma]_G$. We recall Bernstein's construction of a progenerator of $\mathrm{Rep}^{\mathfrak{s}}(G)$ following [53, Section 1]. Let $L^1 = \bigcap_{\nu \in \mathfrak{X}_{\mathrm{nr}}(L)} \mathrm{Ker}\,\nu$, there exists an irreducible supercuspidal subrepresentation σ_1 of $\sigma|_{L^1} = \sigma_1 \oplus \sigma_2 \oplus \ldots \oplus \sigma_r$. Let Σ denote $\mathrm{ind}_{L^1}^L \sigma_1$, we know it is a progenerator of the category $\mathrm{Rep}^{\mathfrak{s}}(L)$. Now consider $\mathrm{i}_P^G(\Sigma)$ for some $P \in \mathcal{P}(L)$. We have (see [53, Section 1.4-1.6])

Theorem 6.5.1. *The isomorphism class of* $\mathrm{i}_P^G(\Sigma)$ *is independent of the parabolic subgroup* $P \in \mathcal{P}(L)$.

Combining with the fact that $\bigoplus_{P \in \mathcal{P}(L)} \mathrm{i}_P^G(\Sigma)$ is a progenerator, we have an equivalence of categories:

Corollary 6.5.2. *For any* $P \in \mathcal{P}(L)$, *the representation* $\mathrm{i}_P^G(\Sigma)$ *is a progenerator of* $\mathrm{Rep}^{\mathfrak{s}}(G)$,

$$\begin{aligned} \mathcal{E}_G : \mathrm{Rep}^{\mathfrak{s}}(G) &\to \mathrm{Mod}\text{-}\mathrm{End}_G(\mathrm{i}_P^G\,\Sigma) \\ V &\mapsto \mathrm{Hom}_G(\mathrm{i}_P^G(\Sigma), V) \end{aligned} \tag{6.5.1}$$

is an equivalence of categories.

We may omit the sub-index of $\mathcal{E}_G$ if no confusion is caused.

Induction and restriction

Let M be a Levi subgroup such that $L \subset M$, $P = LU \in \mathcal{P}(L)$ and $Q = MN \in \mathcal{P}(M)$. The pair (L,σ) determines $\mathfrak{s}_M = [L,\sigma]_M$ and $\mathfrak{s} = [L,\sigma]_G$.

Theorem 6.5.3 (Theorem of section 5, [53]). *The two following diagrams commute up to natural equivalence*

$$\begin{CD} \mathrm{Rep}^{\mathfrak{s}}(G) @>{\mathcal{E}_G}>> \mathrm{Mod}\text{-}\mathrm{End}_G(\mathrm{i}_P^G\,\Sigma) \\ @V{\mathrm{r}_{\bar{U}}^{\mathfrak{s}_M}}VV @VV{\mathrm{Res}^{\mathrm{End}_G(\mathrm{i}_P^G\,\Sigma)}_{\mathrm{End}_M(\mathrm{i}_{M\cap P}^M\,\Sigma)}}V \\ \mathrm{Rep}^{\mathfrak{s}_M}(M) @>>{\mathcal{E}_M}> \mathrm{Mod}\text{-}\mathrm{End}_M(\mathrm{i}_{M\cap P}^M\,\Sigma) \end{CD} \tag{6.5.2}$$

$$\begin{CD} \mathrm{Rep}^{\mathfrak{s}}(G) @>{\mathcal{E}_G}>> \mathrm{Mod}\text{-}\mathrm{End}_G(\mathrm{i}_P^G\,\Sigma) \\ @A{\mathrm{i}_Q^G}AA @AA{\mathrm{Ind}^{\mathrm{End}_G(\mathrm{i}_P^G\,\Sigma)}_{\mathrm{End}_M(\mathrm{i}_{M\cap P}^M\,\Sigma)}}A \\ \mathrm{Rep}^{\mathfrak{s}_M}(M) @>>{\mathcal{E}_M}> \mathrm{Mod}\text{-}\mathrm{End}_M(\mathrm{i}_{M\cap P}^M\,\Sigma) \end{CD} \tag{6.5.3}$$

In the first diagram (6.5.2), $\mathrm{r}_{\bar{U}}^{\mathfrak{s}_M} := \mathrm{pr}_{\mathfrak{s}_M} \circ \mathrm{r}_{L\bar{U}}^G$, $\mathrm{Res}^{\mathrm{End}_G(\mathrm{i}_P^G\,\Sigma)}_{\mathrm{End}_M(\mathrm{i}_{M\cap P}^M\,\Sigma)}$ *is the restriction along the natural ring injection* $\mathrm{End}_M(\mathrm{i}_{M\cap P}^M\,\Sigma) \to \mathrm{End}_G(\mathrm{i}_Q^G(\mathrm{i}_{M\cap P}^M\,\Sigma)) = \mathrm{End}_G(\mathrm{i}_P^G\,\Sigma)$ *provided by the functor* i_Q^G. *In the second diagram (6.5.3),* $\mathrm{Ind}^{\mathrm{End}_G(\mathrm{i}_P^G\,\Sigma)}_{\mathrm{End}_M(\mathrm{i}_{M\cap P}^M\,\Sigma)} = \text{-}\bigotimes_{\mathrm{End}_M(\mathrm{i}_{M\cap P}^M\,\Sigma)} \mathrm{End}_G(\mathrm{i}_P^G\,\Sigma)$.

We now apply this theorem to the opposite parabolic subgroups $\overline{P}$ and $\overline{Q}$, noticing the fact $\mathrm{i}_P^G\,\Sigma \cong \mathrm{i}_{\overline{P}}^G\,\Sigma$ and $\mathrm{i}_{P\cap M}^M\,\Sigma \cong \mathrm{i}_{\overline{P}\cap M}^M\,\Sigma$ from the proof of 6.5.1, we have

$$\begin{CD} \mathrm{Rep}^{\mathfrak{s}}(G) @>{\mathcal{E}_G}>> \mathrm{Mod}\text{-}\mathrm{End}_G(\mathrm{i}_P^G\,\Sigma) \\ @V{\mathrm{r}_{U}^{\mathfrak{s}_M}}VV @VV{\mathrm{Res}^{\mathrm{End}_G(\mathrm{i}_P^G\,\Sigma)}_{\mathrm{End}_M(\mathrm{i}_{M\cap P}^M\,\Sigma)}}V \\ \mathrm{Rep}^{\mathfrak{s}_M}(M) @>>{\mathcal{E}_M}> \mathrm{Mod}\text{-}\mathrm{End}_M(\mathrm{i}_{M\cap P}^M\,\Sigma) \end{CD} \tag{6.5.4}$$

$$\begin{CD} \mathrm{Rep}^{\mathfrak{s}}(G) @>{\mathcal{E}_G}>> \mathrm{Mod}\text{-}\mathrm{End}_G(\mathrm{i}_P^G\,\Sigma) \\ @A{\mathrm{i}_{\overline{Q}}^G}AA @AA{\mathrm{Ind}^{\mathrm{End}_G(\mathrm{i}_P^G\,\Sigma)}_{\mathrm{End}_M(\mathrm{i}_{M\cap P}^M\,\Sigma)}}A \\ \mathrm{Rep}^{\mathfrak{s}_M}(M) @>>{\mathcal{E}_M}> \mathrm{Mod}\text{-}\mathrm{End}_M(\mathrm{i}_{M\cap P}^M\,\Sigma) \end{CD} \tag{6.5.5}$$

Comparing (6.5.3) and (6.5.5), we find

Corollary 6.5.4. *Under the assumptions of this section,* $\mathrm{i}_{\overline{Q}}^G$ *is equivalent to* i_Q^G.

6.6 Comparing the involution with the Aubert-Zelevinsky duality

Let us fix a $\mathfrak{s} = [L, \sigma]_G \in \mathfrak{B}(G)$ with L a standard Levi subgroup, and $P = LU \in \mathcal{P}(L)$. Aubert-Zelevinsky duality $\mathrm{D}_G = \sum_{I \subset S} (-1)^{|I|}\, \mathrm{i}_{P_I}^G \circ \mathrm{r}_{P_I}^G$ acts on $[\pi]$ for $\pi \in \mathrm{Rep}_{\mathrm{f}}^{\mathfrak{s}}(G)$ by the formula (6.2.1). Let π_0 denote an irreducible component of π that is a subquotient of $\mathrm{i}_P^G(\sigma \otimes \nu)$ for some $\nu \in \mathfrak{X}_{\mathrm{nr}}(L)$, then $\mathrm{r}_{P_I}^G(\pi_0)$ is a subquotient of $\mathrm{r}_{P_I}^G(\mathrm{i}_P^G(\sigma \otimes \nu))$. From the [14, Geometric Lemma 2.11], $\mathrm{r}_{P_I}^G \circ \mathrm{i}_P^G$ admits a filtration by functors of the form $\mathrm{i}_{w(L) \cap L_I}^{L_I} \circ w \circ \mathrm{r}_{L \cap w^{-1}(L_I)}^L$ for $w \in W^{L, L_I}$ (see the definition of W^{L, L_I} in *loc.cit.* 2.11). Then $\mathrm{r}_{P_I}^G(\pi_0)$ is contained in subquotients of the sum of $\mathrm{i}_{w(L)}^{L_I}(w(\sigma \otimes \nu))$ for all $w \in W^{L, L_I}$ satisfying $w(L) \subset L_I$. The pair $(w(L), w(\sigma \otimes \nu))$ determines $[w(L), w(\sigma \otimes \nu)]_{L_I} \in \mathfrak{B}(L_I)$ (in general differs from $[L, \sigma]_{L_I}$), by fixing a representative $g \in G$ of w, we see such pair corresponds to $\mathfrak{s}$ in $\mathfrak{B}(G)$: $[{}^{g^{-1}}(w(L)), {}^{g^{-1}}(w(\sigma \otimes \nu))]_G = [L, \sigma]_G \in \mathfrak{B}(G)$.

We denote by $\mathfrak{S}_I := \{\mathfrak{s}_{L_I} = [w(L), w(\sigma)]_{L_I} \mid w \in W^{L, L_I},\ w(L) \subset L_I\}$. We now adapt the diagrams (6.5.4) and (6.5.5) accordingly. Let $M = L_I$, $\mathfrak{s}_{L_I} \in \mathfrak{S}_I$, the progenerator of $\mathrm{Rep}^{[w(L), w(\sigma)]_L}(w(L))$ is $\Sigma_w = \mathrm{ind}_{w(L^1)}^{w(L)}(w(\sigma_1))$. We also have $\mathrm{End}_G(\mathrm{i}_P^G(w(\Sigma))) \cong \mathrm{End}_G(\mathrm{i}_P^G(\Sigma))$ by 6.5.1, the diagrams become

$$\begin{tikzcd}
\mathrm{Rep}^{\mathfrak{s}}(G) \arrow[r, "\mathcal{E}_G"] \arrow[d, "\mathrm{pr}_{L_I}^{\mathfrak{s}_{L_I}} \circ \mathrm{r}_{P_I}^G"'] & \mathrm{Mod}\text{-}\mathrm{End}_G(\mathrm{i}_P^G \Sigma) \arrow[d, "\mathrm{Res}^{\mathrm{End}_G(\mathrm{i}_P^G \Sigma)}_{\mathrm{End}_{L_I}(\mathrm{i}_{L_I \cap P}^{L_I} \Sigma_w)}"] \\
\mathrm{Rep}^{\mathfrak{s}_{L_I}}(L_I) \arrow[r, "\mathcal{E}_{L_I}"'] & \mathrm{Mod}\text{-}\mathrm{End}_{L_I}(\mathrm{i}_{L_I \cap P}^{L_I} \Sigma_w)
\end{tikzcd} \tag{6.6.1}$$

$$\begin{tikzcd}
\mathrm{Rep}^{\mathfrak{s}}(G) \arrow[r, "\mathcal{E}_G"] & \mathrm{Mod}\text{-}\mathrm{End}_G(\mathrm{i}_P^G \Sigma) \\
\mathrm{Rep}^{\mathfrak{s}_{L_I}}(L_I) \arrow[r, "\mathcal{E}_{L_I}"'] \arrow[u, "\mathrm{i}_{P_I}^G"] & \mathrm{Mod}\text{-}\mathrm{End}_{L_I}(\mathrm{i}_{L_I \cap P}^{L_I} \Sigma_w) \arrow[u, "\mathrm{Ind}^{\mathrm{End}_G(\mathrm{i}_P^G \Sigma)}_{\mathrm{End}_{L_I}(\mathrm{i}_{L_I \cap P}^{L_I} \Sigma_w)}"']
\end{tikzcd} \tag{6.6.2}$$

where pr_{L_I} is define in (6.3.3). We have shown in the above discussion that the Bernstein blocks $\mathrm{Rep}^{\mathfrak{s}_{L_I}}(L_I)$ with $\mathrm{pr}_{L_I}^{\mathfrak{s}_{L_I}} \circ \mathrm{r}_{P_I}^G(\pi_0) \neq 0$ are all indexed by some $\mathfrak{s}_{L_I} \in \mathfrak{S}_I$, thus the functor $\mathrm{r}_{P_I}^G$ restricted to $\mathrm{Rep}^{\mathfrak{s}}(G)$ can be written as a sum $\sum_{\mathfrak{s}_{L_I} \in \mathfrak{S}_I} \mathrm{pr}_{L_I}^{\mathfrak{s}_{L_I}} \circ \mathrm{r}_{P_I}^G$. The duality functor has the form

$$\mathrm{D}_G = \sum_{I \subset S} (-1)^{|I|} \Big(\sum_{\mathfrak{s}_{L_I} \in \mathfrak{S}_I} \mathrm{i}_{P_I}^G \circ \mathrm{pr}^{\mathfrak{s}_{L_I}} \circ \mathrm{r}_{P_I}^G \Big).$$

By [57], we know all these endomorphism algebras are generalized (in the sense twisted with 2-cocycle) affine Hecke algebras. On the Hecke algebra side, the involution is

$$\mathrm{D}_{\mathcal{H}} = \sum_{I \subset S} (-1)^{|I|} \Big(\sum_{\mathfrak{s}_{L_I} \in \mathfrak{S}_I} \mathrm{Ind}_{\mathrm{End}_{L_I}(\mathrm{i}_{L_I \cap P}^{L_I} \Sigma_w)}^{\mathrm{End}_G(\mathrm{i}_P^G \Sigma)} \circ \mathrm{Res}_{\mathrm{End}_{L_I}(\mathrm{i}_{L_I \cap P}^{L_I} \Sigma_w)}^{\mathrm{End}_G(\mathrm{i}_P^G \Sigma)} \Big). \tag{6.6.3}$$

We mention in the following that:

1. If the supercuspidal support is $(T, 1)$, in this case $\mathfrak{S}_I = \{\mathfrak{s}_{L_I} = [T, 1]_{L_I}\}$ is a singleton, $\mathrm{r}_{P_I}^G : \mathrm{Rep}^{\mathfrak{s}}(G) \to \mathrm{Rep}^{\mathfrak{s}_{L_I}}(L_I)$ is the restriction of $\mathrm{r}_{P_I}^G$ to $\mathrm{Rep}^{\mathfrak{s}}(G)$. The involution on the Hecke algebra side becomes (4.0.1) in the Chapter 4 after passing to the category of left modules of the opposite algebras.
2. In the finite group of Lie type cases, we interpreted the left hand side of (5.7.1) as the counterpart on the "finite Hecke algebra side" of Alvis-Curtis-Kawanaka duality. The underlying ideas inspire the approach that we take in this Section to get (6.6.3) which is introduced as the counterpart on the "generalized affine Hecke algebra side" of Aubert-Zelevinsky duality.

Chapter 7

Computations for the p-adic group G_2

This section is based on [10] and [9]. We use notations from section 6.1. Let $\mathbb{G}$ be the exception group of type G_2, and $G = \mathbb{G}(F)$ the F-rational points. Let T be a maximal F-split torus of G_2 and $B = TU$ be a Borel subgroup. In this section, we will compute the restriction of the Aubert-Zelevinsky duality D_G to $\mathrm{Rep}_{\mathrm{f}}^{\mathfrak{s}}(G)$ with $\mathfrak{s} = [L, \sigma]_G$ where L is a proper subgroup of G_2 and is thus isomorphic to $\mathrm{GL}_2(F)$ or T. By the discussion in section 6.5, the equivalence of categories will also give us the involution on the category of finite dimensional right modules over $\mathcal{H}^{\mathfrak{s}}(G) := \mathrm{End}_G(\mathrm{i}_P^G(\Sigma))$. We will use the notations of indexing triples and standard modules from [44]. From section 7.4 to section 7.10.2 are the case-by-case discussions following the methods that first appear in [47]. We summarize the results in Section 7.1.

7.1 Main results

We list the main results in the following tables:

$\mathfrak{s} = [M_\alpha, \sigma]$	$\mathrm{Rep}^s(G_2)$	$\mathcal{H}^{\mathfrak{s}}$- mod
$(-)$	$\pi(\sigma)$	$M_{t_a, e_{\alpha_1}, 1}$
$D(-)$	$-J(\sigma)$	$-M_{t_a, 0, 1}$

Table 7.1

$\mathfrak{s} = [M_\beta, \sigma \otimes \tilde{\beta}]$	$\mathrm{Rep}^s(G_2)$	$\mathcal{H}^{\mathfrak{s}}$- mod
$(-)$	$\pi(\sigma)$	$M_{t_a, e_{\alpha_1}, 1}$
$D(-)$	$-J(\sigma)$	$-M_{t_a, 0, 1}$

Table 7.2

$\mathfrak{s} = [T, \xi \otimes 1]_G$	Case (3) ξ not quadratic, $\mathfrak{J}^{\mathfrak{s}} = \mathrm{GL}_2(\mathbb{C})$	
	$\mathrm{Rep}^s(G_2)$	$\mathcal{H}^{\mathfrak{s}}$- mod
$(-)$	$I_\alpha\left(\delta(\nu^{\pm 1/2}\xi_2)\right)$	$M_{t_a,e_\alpha,1}$
$D(-)$	$I_\alpha\left(\nu^{\pm 1/2}\xi_2 \circ \det\right)$	$M_{t_a,0,1}$

Table 7.3

$\mathfrak{s} = [T, \xi \otimes \xi]$	Case (2) with ξ_2 ramified cubic, $\mathfrak{J}^{\mathfrak{s}} = \mathrm{SL}_3(\mathbb{C})$		Case (2) with ξ_2 ramified noncubic, $\mathfrak{J}^{\mathfrak{s}} = \mathrm{GL}_2(\mathbb{C})$		ξ_2 is unramified, $s_2 \neq \pm 1$, $\mathfrak{J}^{\mathfrak{s}} = G_2(\mathbb{C})$	
	$\mathrm{Rep}^s(G_2)$	$\mathcal{H}^{\mathfrak{s}}$- mod	$\mathrm{Rep}^s(G_2)$	$\mathcal{H}^{\mathfrak{s}}$- mod	$\mathrm{Rep}^s(G_2)$	$\mathcal{H}^{\mathfrak{s}}$- mod
$(-)$	$\pi(\nu^{\mp 1} \otimes \xi_2)$	$M_{t_b,e_{\alpha_2},1}$	$\pi(\nu^{\mp 1} \otimes \xi_2)$	$M_{t_a,e_{\alpha_1},1}$	$\pi(\nu^{\mp 1} \otimes \xi_2)$	$M_{t_g,e_{\alpha_1},1}$
$D(-)$	$J(\nu^{\mp 1} \otimes \xi_2)$	$M_{t_b,0,1}$	$J(\nu^{\mp 1} \otimes \xi_2)$	$M_{t_a,0,1}$	$J(\nu^{\mp 1} \otimes \xi_2)$	$M_{t_g,0,1}$

Table 7.4

Case (3) with χ ramified quadratic, $\mathfrak{J}^{\mathfrak{s}} = \mathrm{SO}_4(\mathbb{C})$				Case (3) with χ ramified cubic, $\mathfrak{J}^{\mathfrak{s}} = \mathrm{SL}_3(\mathbb{C})$			
$\mathrm{Rep}^s(G_2)$	$\mathcal{H}^{\mathfrak{s}}$- mod	$\mathrm{Rep}^s(G_2)$	$\mathcal{H}^{\mathfrak{s}}$- mod	$\mathrm{Rep}^s(G_2)$	$\mathcal{H}^{\mathfrak{s}}$- mod	$\mathrm{Rep}^s(G_2)$	$\mathcal{H}^{\mathfrak{s}}$- mod
$\pi(\chi)$	$M_{(t_a,e_{\alpha_1},1),(t_a,e_{\alpha_1},1)}$	$J_\alpha(1/2, \delta(\chi))$	$M_{t_a,e_{\alpha_1},1}$	$\pi(\chi)$	$M_{t_a,e_{\alpha^\vee}+e_{2\alpha^\vee+3\beta^\vee},1}$	$J_\alpha(1/2, \delta(\chi))$	$M_{t_a,e_{\alpha^\vee},1}$
$J_\beta(1, \pi(1,\chi))$	$M_{(t_a,0,1),(t_a,0,1)}$	$J_\beta(1/2, \delta(\chi))$	$M_{(t_a,e_{\alpha_1},1),(t_a,0,1)}$	$J_\beta(1, \pi(\chi^{-1}, \chi^{-1}))$	$M_{t_a,0,1}$	$J_\alpha(1/2, \delta(\chi^{-1}))$	$M_{t_a,e_{3\beta^\vee+2\alpha^\vee},1}$

7.4 Continued

$\mathfrak{s} = [T, \otimes 1]$	Case (2) with $\chi = 1$, $s = 1/2$, $\mathfrak{J}^{\mathfrak{s}} = G_2(\mathbb{C})$					
	$\mathrm{Rep}^s(G_2)$	$\mathcal{H}^{\mathfrak{s}}$- mod	$\mathrm{Rep}^s(G_2)$	$\mathcal{H}^{\mathfrak{s}}$- mod	$\mathrm{Rep}^s(G_2)$	$\mathcal{H}^{\mathfrak{s}}$- mod
$(-)$	$\pi(1)$	$M_{t_e,e_{\alpha^\vee}+e_{\alpha^\vee+2\beta^\vee},(21)}$	$J_\beta(1/2, \delta(1))$	$M_{t_e,e_{\alpha^\vee+\beta^\vee},1}$	$\pi'(1)$	$M_{t_e,e_{\alpha^\vee}+e_{\alpha^\vee+2\beta^\vee},(3)}$
$D(-)$	$J_\alpha(1/2, \delta(1))$	$M_{t_e,e_{\alpha^\vee},1}$	$J_\beta(1/2, \delta(1))$	$M_{t_e,e_{\alpha^\vee+\beta^\vee},1}$	$J_\beta(1, \pi(1,1))$	$M_{t_e,0,1}$

Table 7.5

Case (2) with $s = 3/2$ and $\chi = 1$				Case (3) with ξ_2 unramified, $\mathfrak{J}^{\mathfrak{s}} = G_2(\mathbb{C})$	
$\mathrm{Rep}^s(G_2)$	$\mathcal{H}^{\mathfrak{s}}$- mod	$\mathrm{Rep}^s(G_2)$	$\mathcal{H}^{\mathfrak{s}}$- mod	$\mathrm{Rep}^s(G_2)$	$\mathcal{H}^{\mathfrak{s}}$- mod
St_{G_2}	$M_{t_a,e_{\alpha^\vee}+e_{\beta^\vee},1}$	$J_\alpha(3/2, \delta(1))$	$M_{t_a,e_{\alpha^\vee},1}$	$I_\alpha\left(\delta(\nu^{\pm 1/2}\xi_2)\right)$	$M_{t_g,e_\alpha,1}$
1_{G_2}	$M_{t_a,0,1}$	$J_\beta(5/2, \delta(1))$	$M_{t_a,e_{\beta^\vee},1}$	$I_\alpha\left(\nu^{\pm 1/2}\xi_2 \circ \det\right)$	$M_{t_g,0,1}$

7.5 Continued

Case (3) with χ unramified quadratic, $\mathfrak{J}^{\mathfrak{s}} = \mathbb{G}_2(\mathbb{C})$				Case (3) with $s = 1/2$ and χ cubic, $\mathfrak{J}^{\mathfrak{s}} = \mathbb{G}_2(\mathbb{C})$			
$\mathrm{Rep}^s(G_2)$	$\mathcal{H}^{\mathfrak{s}}$- mod	$\mathrm{Rep}^s(G_2)$	$\mathcal{H}^{\mathfrak{s}}$- mod	$\mathrm{Rep}^s(G_2)$	$\mathcal{H}^{\mathfrak{s}}$- mod	$\mathrm{Rep}^s(G_2)$	$\mathcal{H}^{\mathfrak{s}}$- mod
$\pi(\chi)$	$M_{t_d, e_{\alpha^\vee} + e_{\alpha^\vee + 2\beta^\vee}, 1}$	$J_\alpha(1/2, \delta(\chi))$	$M_{t_d, e_{\alpha^\vee}, 1}$	$\pi(\chi)$	$M_{t_c, e_{\alpha^\vee} + e_{\alpha^\vee + 3\beta^\vee}, 1}$	$J_\alpha(1/2, \delta(\chi))$	$M_{t_c, e_{\alpha^\vee}, 1}$
$J_\beta(1, \pi(1, \chi))$	$M_{t_d, 0, 1}$	$J_\beta(1/2, \delta(\chi))$	$M_{t_d, e_{2\beta^\vee + \alpha^\vee}, 1}$	$J_\beta(1, \pi(\chi^{-1}, \chi^{-1}))$	$M_{t_c, 0, 1}$	$J_\alpha(1/2, \delta(\chi^{-1}))$	$M_{t_c, e_{3\beta^\vee + \alpha^\vee}, 1}$

7.5 Continued II

7.2 Principal blocks

We recall here some results from [52] section 6 to 8. Let T^1 be the unique maximal compact subgroup of T and $\chi : T^1 \to \mathbb{C}$ be a smooth character. Let $\widetilde{\chi}$ be any character of T extending χ, the inertial equivalence class $\mathfrak{s}_\chi := [T, \widetilde{\chi}]$ depends only on χ (it is well-defined). An $\mathfrak{s}$-*type* in G is a pair (K, ρ) where K is a compact open subgroup of G and ρ is an irreducible smooth representation of K such that an irreducible representation π of G contains ρ if and only if $\mathfrak{I}(\pi) = \mathfrak{s}$. In [52], A. Roche constructed an $\mathfrak{s}_\chi$-type (J, ρ) where J is a compact open subgroup of G and ρ is a smooth character of J such that $J \cap T = T^1$ and $\rho|_{J \cap T} = \chi$. Recall Definition 6.4.1, $\mathcal{H}(G, \rho)$ is the space of complex compactly supported functions equipped with a star operation κ given by

$$\kappa(f)(x) = \overline{f(x^{-1})}, \text{ for } f \in \mathcal{H}(G, \rho).$$

This is the same κ as we introduced in Definition 4.4.2, see [12, Section 5] for more details. There is an equivalence of categories $\mathrm{Rep}^{\mathfrak{s}_\chi}(G) \to \mathcal{H}(G, \rho)$- Mod by *loc.cit.* Theorem 7.5 and Corollary 7.9.

There exists a dual group interpretation of $\mathcal{H}(G, \rho)$. Applying Local Langlands correspondence for T, we have a homomorphism $\varphi_{\widetilde{\chi}} : W_F \to \widehat{T}$ associated with $\widetilde{\chi} : T \to \mathbb{C}$. By local class field theory, $\varphi_{\widetilde{\chi}}|_{I_F}$ depends only on χ and we denote it by φ_χ. Let $\mathfrak{J}^{\mathfrak{s}}$ denote the centralizer in $\widehat{G}$ of the image of φ_χ. When $\mathbb{G}$ has connected center, the group $\mathfrak{J}^{\mathfrak{s}}$ is connected and its Weyl group is isomorphic to the group $W^{\mathfrak{s}}$. Let $J_{\mathfrak{s}}$ denote the group of F-rational points of the F-split reductive group whose Langlands dual is the group $\mathfrak{J}^{\mathfrak{s}}$. By [52, section 8], the algebra $\mathcal{H}(G, \rho)$ and $\mathcal{H}(J_{\mathfrak{s}}, 1_J)$ are isomorphic via a family of κ-preserving, support-preserving isomorphisms.

We recall here some results from [44] about indexing triples and standard modules for affine Hecke algebras. Let $\mathcal{G}$ be a simple complex algebraic group with root system R and weight lattice X. Recall that q is the number of elements in the residue field of F. *An indexing triple* (s, n, ρ) consists of a semisimple element $s \in \mathcal{G}$, a nilpotent element $n \in \mathrm{Lie}(\mathcal{G})$ such that $\mathrm{Ad}(s)n = q^2 n$ and an irreducible representation of the component group $A(s, n) := Z_{\mathcal{G}}(s, n)/Z_{\mathcal{G}}(s, n)^\circ$ where $Z_{\mathcal{G}}(s, n) := Z_{\mathcal{G}}(s) \cap Z_{\mathcal{G}}(n)$. Let $n = \exp(n)$ denote the corresponding unipotent element of $\mathcal{G}$. Let $K(\mathcal{B}_{s,u})$ be the K-theory of the variety $\mathcal{B}_{s,u}$ where $\mathcal{B}_{s,u}$ is the variety of Borel subgroups of $\mathcal{G}$ containing both s and u.

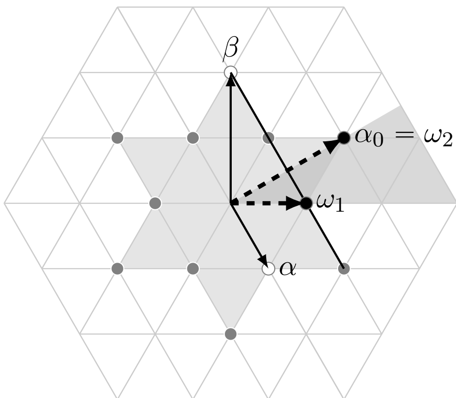
The group $A(s,u)$ and $\mathcal{H}(G,\rho)$ act on $K(\mathcal{B}_{s,u})$ and their actions commute. The standard modules $M_{s,n,\rho}$ are the $\mathcal{H}(G,\rho)$-modules in the decomposition:

$$K(\mathcal{B}_{s,u}) = \bigoplus_{\rho\in\mathrm{Irr}(A(s,u))} M_{s,n,\rho}\otimes\rho. \tag{7.2.1}$$

7.3 Preliminaries on G_2

7.3.1 Some structure theory of G_2

Let V be the hyperplane of $\mathbb{R}^3 = \mathrm{Span}_{\mathbb{R}}(e_1,e_2,e_3)$ formed by points whose sum of coordinates is zero.



The choice of simple roots $\alpha = \dfrac{1}{\sqrt{3}}(e_1-e_2)$ (*the short root*) and $\beta = \dfrac{1}{\sqrt{3}}(-2e_1+e_2+e_3)$ (*the long root*) define a basis for the root system R of G, and $R^+ = \{\alpha,\beta,\alpha+\beta,2\alpha+\beta,3\alpha+\beta,3\alpha+2\beta\}$. The dominant weights are $\omega_1 = 2\alpha+\beta = \dfrac{1}{\sqrt{3}}(-e_2+e_3)$ and $\omega_2 = 3\alpha+2\beta = \dfrac{1}{\sqrt{3}}(-e_1-e_2+2e_3)$. For any $\gamma\in R$, we denote by $\gamma^\vee$ the corresponding coroot, and s_γ the reflection in W defined by γ. Let $B = TU$ be the corresponding Borel subgroup, P_γ be the maximal standard parabolic subgroup of G generated by B and $x_{\pm\gamma}(F)$ (defined in [9, Section 4]).

We need to clarify some notations and definitions on Levi subgroups of G_2, following in A-M. Aubert & Y. Xu [9] and G. Muic [47], we define:

$$\begin{aligned}(t_1,t_2)\in T\cong F^\times\times F^\times &\mapsto \begin{pmatrix} t_1 & \\ & t_2\end{pmatrix}\in T_\alpha\\ (t_1,t_2)\in T\cong F^\times\times F^\times &\mapsto \begin{pmatrix} t_2 & \\ & t_1t_2^{-1}\end{pmatrix}\in T_\beta\end{aligned} \tag{7.3.1}$$

as the isomorphisms between the split torus of maximal Levis associated with $\{\alpha,\beta\}$ and T.

Accordingly, if $\chi_1 \otimes \chi_2$ is a character of $F^\times \times F^\times$, then:

$$\begin{aligned} \chi_1 \otimes \chi_2 &\mapsto \begin{pmatrix} \chi_1 & \\ & \chi_2 \end{pmatrix} \text{ as character of } T_\alpha \\ \chi_1 \otimes \chi_2 &\mapsto \begin{pmatrix} \chi_1\chi_2 & \\ & \chi_1 \end{pmatrix} \text{ as character of } T_\beta \end{aligned} \tag{7.3.2}$$

Remark. The notations mean the following:

$$\begin{pmatrix} \chi_1 & \\ & \chi_2 \end{pmatrix} \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} = \chi_1(t_1)\chi_2(t_2) = \begin{pmatrix} \chi_1\chi_2 & \\ & \chi_1 \end{pmatrix} \begin{pmatrix} t_2 & \\ & t_1 t_2^{-1} \end{pmatrix}.$$

The above isomorphisms between T and T_γ extends to isomorphisms between $\mathrm{GL}_2(F)$ and L_γ, for $\gamma = \alpha$ or β. The Weyl group action of characters on $F^\times \times F^\times$ is described in [9, Table 1].

7.3.2 Some definitions and abbreviations

We adapt the definitions introduced in Section 6.1 to the G_2 case. We will frequently use the following fact implicitly: for a smooth representation ρ of a Levi subgroup $L \subset P = LU$, ρ extends to a representation of P by letting U act trivially (and is thus seen as a representation of P). We use ν to denote the normalized absolute value of F. The following functors are abbreviated as follows (with γ equals α or β): the normalized parabolic induction $\mathrm{i}_{T(U\cap L_\gamma)}^{L_\gamma}$ to the intermediate Levi L_γ is abbreviated as I^γ; the normalized Jacquet functor $\mathrm{r}_{P_\gamma}^{G_2}$ from G_2 to P_γ is abbreviated as r_γ. The normalized parabolic induction i_B^G is abbreviated as I; the normalized Jacquet functor $\mathrm{r}_B^{G_2}$ from G_2 to B is abbreviated as $r_\varnothing$. We have $I^\gamma\left(s_\gamma(\chi_1 \otimes \chi_2)\right) = I^\gamma(\chi_1 \otimes \chi_2)$ in $\mathrm{R}(L_\gamma)$ where $\gamma = \alpha,\ \beta$ and s_γ the corresponding simple reflection in the Weyl group. For an admissible representation π of L_γ, $s \in \mathbb{C}$ and $\eta \in X_*(L_\gamma) \otimes_{\mathbb{Z}} \mathbb{R}$ (where $X_*(L_\gamma)$ is the group of F-rational characters of L_γ), we write

$$I_\gamma(\chi, \pi) := \mathrm{i}_{P_\gamma}^G((\nu \circ \chi) \otimes \pi),$$

and

$$I_\gamma(s, \pi) := \mathrm{i}_{P_\gamma}^G((\nu^s \circ \det) \otimes \pi),\ I_\gamma(\pi) := I_\gamma(0, \pi).$$

When π is a tempered irreducible representation and $s \in \mathbb{R}^{>0}$, the representation $I^\gamma(s, \pi)$ has a unique irreducible quotient denoted by $J_\gamma(s, \pi)$ by the Langlands quotient Theorem(see [40, Theorem 3.5]).

We end an equation with $[\mathrm{R}(G)]$ to mean that this is an equation in the Grothendieck group of G.

7.3.3 Representation theory of L_γ, $\gamma = \alpha$ or β

We recall some GL(2) theory since the Levi factors of maximal parabolic subgroups of G_2 are isomorphic to $\mathrm{GL}_2(F)$. (For this subsection only) Denote by $B = TU$ a Borel subgroup of $\mathrm{GL}_2(F)$ with maximal torus T. For a smooth character χ of $F^\times$ and any smooth admissible representation π of $\mathrm{GL}_2(F)$, we denote by $\pi\chi$ the twist of π by $\chi \circ \det$. We define the *generalized Steinberg representation* $\delta(\chi)$ as the unique irreducible subrepresentation of $\mathrm{i}_B^{\mathrm{GL}_2}(\nu^{1/2}\chi \otimes \nu^{-1/2}\chi)$, and the *generalized trivial representation* $\chi \circ \det$ as the unique irreducible quotient of $\mathrm{i}_B^{\mathrm{GL}_2}(\nu^{1/2}\chi \otimes \nu^{-1/2}\chi)$. For unitary characters χ_1, χ_2, denote by $\pi(\chi_1, \chi_2) := I^\gamma(\chi_1 \otimes \chi_2)$ for $\gamma = \alpha$ or β, which is a tempered irreducible representation. Now we recall well-known facts about principal series representations of Levi factors of maximal parabolic subgroups of G_2 following [47, Proposition 1.1].

Proposition 7.3.1. *Suppose that χ, χ_1 and χ_2 are characters of $F^\times$, and $\gamma \in \{\alpha, \beta\}$. Then we have the following.*

(1) The principal series $I^\gamma(\chi_1 \otimes \chi_2)$ of L_γ reduces if and only if $(\chi_1 \otimes \chi_2) \circ \gamma^\vee = \nu^{\pm 1}$. If $I^\gamma(\chi_1 \otimes \chi_2)$ is irreducible, it is isomorphic to $I^\gamma(s_\gamma(\chi_1 \otimes \chi_2))$.

*(2) The principal series $I^\alpha\left(\nu^{1/2}\chi \otimes \nu^{-1/2}\chi\right)$ (*resp. *$I^\alpha\left(\nu^{-1/2}\chi \otimes \nu^{1/2}\chi\right)$) contains $\delta(\chi)$ and $\chi \circ$ det as a unique irreducible subrepresentation (*resp. *quotient) and quotient (*resp. *subrepresentation).*

(3) The principal series $I^\beta\left(\nu^{-1/2}\chi \otimes \nu\right)$ (resp. *$I^\beta\left(\nu^{1/2}\chi \otimes \nu^{-1}\right)$) contains $\delta(\chi)$ and $\chi \circ$* det *as a unique irreducible subrepresentation (*resp. *quotient) and quotient (*resp. *subrepresentation).*

The restriction operators applied to the generalized Steinberg and generalized trivial representations are also adapted depending on $\gamma = \alpha$ or β, we have

$$r_T^{L_\beta}(\nu^s \delta(\chi)) = \nu^{s-1/2}\chi \otimes \nu,$$

and

$$r_T^{L_\beta}(\nu^s \chi \circ \det) = \nu^{s+1/2}\chi \otimes \nu^{-1}.$$

The tensor operation also needs to be adapted depending on γ, we have

$$\nu^s \otimes I^\alpha(\chi_1 \otimes \chi_2) = I^\alpha(\nu^s\chi_1 \otimes \nu^s\chi_2),$$

and

$$\nu^s \otimes I^\beta(\chi_1 \otimes \chi_2) = I^\beta(\chi_1\nu^s \otimes \chi_2).$$

7.4 Case by case discussion

Let $\mathfrak{s} = [L, \sigma]_G$, where L is a proper Levi subgroup of G_2. Let σ be an irreducible supercuspidal representation of L. The Levi subgroups of G_2 are isomorphic to either $\mathrm{GL}_2(F)$ or $F^\times \times F^\times$.

7.5 The intermediate case $\mathfrak{s} = [L_\alpha, \sigma]_G$ and $\mathfrak{s} = [L_\beta, \sigma]_G$

Let ω_σ denote the central character of σ. If $L = L_\alpha$ by [55, Proposition 6.2] and [9, Section 8]:

(1) When $\omega_\sigma \neq 1, \mathrm{i}_P^G(\sigma)$ is reducible and there are no complementary series.

(2) When $\omega_\sigma = 1, \mathrm{i}_P^G(\sigma)$ is irreducible, and $\mathrm{i}_P^G(\sigma \otimes \nu^s)$ is irreducible unless $s = \pm 1/2$.

When $\mathrm{i}_P^G\left(\sigma \otimes \nu^{1/2}\right)$ is reducible, it has has length 2: it has a unique generic discrete series subrepresentation $\pi(\sigma)$ and a unique irreducible pre-unitary non-tempered Langlands quotient, $J(\sigma)$. From [10, 4.1.2.], we know the Hecke algebra $\mathcal{H}^{\mathfrak{s}}$ in the short root case is either

$$\mathcal{H}\left(\{1, s_{3\alpha+2\beta}\}, q_F\right) \ltimes \mathbb{C}[\mathcal{O}]$$

(q_F =number of elements of the residue field) or

$$\mathcal{H}\left(\{1, s_{3\alpha+2\beta}\}, 1\right) \ltimes \mathbb{C}[\mathcal{O}].$$

If $M = L_\beta$, from *loc.cit.* 4.1.1. there are four possibilities for the series and Hecke algebra depending on the Plancherel measure. The reducible case is $I_\beta(\tilde{\beta}, \sigma)$ with $\tilde{\beta} = \langle \rho, \beta \rangle^{-1} \rho$ where ρ is half sum of positive roots. In this case $I_\beta(\tilde{\beta}, \sigma)$ is length 2 with a unique χ-generic subrepresentation $\pi(\sigma)$ (*i.e.* it can be realized on a space of smooth functions W satisfying $W(um) = \chi(u)W(m)$, for $m \in L_\beta$, $u \in U_\beta$, and χ is some generic character of U_β) which is in the discrete series and a preunitary non-tempered quotient $J(\sigma)$. From [10, 4.1.1.], we know the Hecke algebra $\mathcal{H}^{\mathfrak{s}}$ in the long root case are

$$\mathcal{H}\left(\{1, s_{2\alpha+\beta}\}, q_s\right) \ltimes \mathbb{C}[\mathcal{O}]$$

where the parameters have four cases to discuss depending on ω.

In both the short root case and the long root case, we have

$$\mathrm{D}_G(\pi(\sigma)) = -J(\sigma).$$

From now on, we will use the notations of indexing triples and standard modules (7.2.1) (see [44] for more details) to denote the irreducible modules of Hecke algebras. From [9, table 11], we know the involution for the modules of Hecke algebra is

$$D_{\mathcal{H}^{\mathfrak{s}}}([M_{t_a,e_{\alpha_1},1}]) = -M_{t_a,0,1}.$$

7.6 Principal series

We now deal with principal series. We start by recalling the setup in [9, Section 9.3]. We may write $\sigma = \xi_1 \otimes \xi_2 = \chi_1\xi \otimes \chi_2\xi$, with ξ a ramified character of $F^\times$, and unramified characters $\chi_1 = \nu^{s_1}$ and $\chi_2 = \nu^{s_2}$ for $s_1, s_2 \in \mathbb{C}$. We start with the assumption that at least one of $\chi_1\xi$ and $\chi_2\xi$ is non-unitary, then following [47, Proposition 3.1], $I(\nu^{s_1}\xi_1 \otimes \nu^{s_2}\xi_2)$ is irreducible unless:

(1) $s_1 \pm 1$, $\xi = 1$, s_2 arbitrary. (*resp.* $s_2 \pm 1$, $\xi = 1$, s_1 arbitrary)

(2) $\nu^{s_1+s_2}\xi^2 = \nu^{\pm 1}$,

(3) $s_1 - s_2 = \pm 1$,

(4) $\nu^{2s_1+s_2}\xi^3 = \nu^{\pm 1}$,

(5) $\nu^{s_1+2s_2}\xi^3 = \nu^{\pm 1}$.

The case (1) and (2) are in fact equivalent, (3) and (4) are equivalent.

7.6.1 Case (2) within the case $\mathfrak{s} = [T,\ \xi \otimes \xi]_G$

In case (2), we have $\nu_F^{s_1}\xi = \nu^{-s_2 \pm 1}\xi^{-1}$. By [15, Lemma 5.4 (iii)], $I(\xi_1 \otimes \xi_2) = I(\nu^{-s_2\pm 1}\xi^{-1} \otimes \nu^{s_2}\xi) = I({}^{w}(\nu^{-s_2\pm 1}\xi^{-1} \otimes \nu^{s_2}\xi)) = I(\nu^{\mp 1} \otimes \xi_2)$ [R(G)], where $w = s_\alpha s_\beta s_\alpha s_\beta s_\alpha$.

For the principal series case, by Section 7.2, we have the Hecke algebras $\mathcal{H}^{\mathfrak{s}} = \mathcal{H}(J_{\mathfrak{s}}, 1_J)$, we only need to find the $J_{\mathfrak{s}}$ or their Langlands dual groups.

Now we discuss the cases when $I^\alpha(\nu^{\mp 1} \otimes \xi_2\nu)$ and $I^\beta(\nu^{\mp 1} \otimes \xi_2\nu)$ are irreducible. If $\nu^{\mp 1} \otimes \xi_2 \notin \{\nu \otimes 1,\ \nu \otimes \nu^2,\ \nu^{-1} \otimes 1,\ \nu^{-1} \otimes \nu^{-2},\ \nu^2 \otimes \nu^{-1},\ 1 \otimes \nu^{-1},\ 1 \otimes \nu,\ \nu^{-2} \otimes \nu\}$, then $I^\alpha(\nu^{\mp 1 - s} \otimes \xi_2\nu^{-s})$ and $I^\beta(\nu^{\mp 1} \otimes \xi_2\nu)$ are irreducible, but $I(\nu^{\pm 1} \otimes \xi_2)$ is reducible.

Lemma 7.6.1. *Under the above assumption, applying r_α to $I(\nu^{\mp 1} \otimes \xi_2)$, we have the following*

$$\begin{aligned} r_\alpha I(\nu^{\mp 1} \otimes \xi_2) = {} & I^\alpha(\nu^{\mp 1} \otimes \xi_2) + I^\alpha(\xi_2^{-1} \otimes \nu^{\pm 1}) + I^\alpha(\nu^{\mp 1}\xi_2 \otimes \xi_2^{-1}) \\ & + I^\alpha(\nu^{\mp 1}\xi_2 \otimes \nu^{\pm 1}) + I^\alpha(\nu^{\mp 1} \otimes \nu^{\pm 1}\xi_2^{-1}) + I^\alpha(\xi_2 \otimes \nu^{\pm 1}\xi_2^{-1}). \end{aligned} \tag{7.6.1}$$

Proof. We know that $r_\alpha(I(\nu^{\mp 1} \otimes \xi_2)) = r_\alpha(I_\alpha(I^\alpha(\nu^{\mp 1} \otimes \xi_2)))$. Applying [14] to the functor $r_\alpha \circ I_\alpha$, we have:

$$\begin{aligned} r_\alpha I(\nu^{\mp 1} \otimes \xi_2) = {} & I^\alpha(\nu^{\mp 1} \otimes \xi_2) + s_{3\alpha+2\beta} \circ I^\alpha(\nu^{\mp 1} \otimes \xi_2) + I^\alpha \circ s_\beta \circ r_T^{L_\alpha}(I^\alpha(\nu^{\mp 1} \otimes \xi_2)) \\ & + I^\alpha \circ s_{\alpha+\beta} \circ r_T^{L_\alpha}(I^\alpha(\nu^{\mp 1} \otimes \xi_2)). \end{aligned} \tag{7.6.2}$$

The same theorem applied to $r_T^{L_\alpha} \circ I^\alpha$ above, we obtain that

$$r_T^{L_\alpha}(I^\alpha(\nu^{\mp 1} \otimes \xi_2)) = \nu^{\mp 1} \otimes \xi_2 + \xi_2 \otimes \nu^{\mp 1}. \tag{7.6.3}$$

Thus the following holds

$$\begin{aligned} r_\alpha I(\nu^{\mp 1} \otimes \xi_2) = {} & I^\alpha(\nu^{\mp 1} \otimes \xi_2) + I^\alpha(\xi_2^{-1} \otimes \nu^{\pm 1}) + I^\alpha(\nu^{\mp 1}\xi_2 \otimes \xi_2^{-1}) \\ & + I^\alpha(\nu^{\mp 1}\xi_2 \otimes \nu^{\pm 1}) + I^\alpha(\nu^{\mp 1} \otimes \nu^{\pm 1}\xi_2^{-1}) + I^\alpha(\xi_2 \otimes \nu^{\pm 1}\xi_2^{-1}). \end{aligned} \tag{7.6.4}$$

□

We deduce from the above Lemma that

$$\begin{aligned} r_\varnothing I(\nu^{\mp 1} \otimes \xi_2) = {} & \nu^{\mp 1} \otimes \xi_2 + \xi_2 \otimes \nu^{\mp 1} + \xi_2^{-1} \otimes \nu^{\pm 1} + \nu^{\pm 1} \otimes \xi_2^{-1} + \nu^{\mp 1}\xi_2 \otimes \xi_2^{-1} \\ & + \xi_2^{-1} \otimes \nu^{\mp 1}\xi_2 + \nu^{\mp 1}\xi_2 \otimes \nu^{\pm 1} + \nu^{\pm 1} \otimes \nu^{\mp 1}\xi_2 + \nu^{\mp 1} \otimes \nu^{\pm 1}\xi_2^{-1} \\ & + \nu^{\pm 1}\xi_2^{-1} \otimes \nu^{\mp 1} + \xi_2 \otimes \nu^{\pm 1}\xi_2^{-1} + \nu^{\pm 1}\xi_2^{-1} \otimes \xi_2. \end{aligned} \tag{7.6.5}$$

By [54], we know $I(\nu^{\mp 1} \otimes \xi_2)$ has length $2 \times \mathrm{Card}(\{s_{3\alpha+2\beta}\}) = 2$. Let $\pi(\nu^{\mp 1} \otimes \xi_2)$ denote its irreducible subrepresentation and $J(\nu^{\mp 1} \otimes \xi_2)$ denote its irreducible quotient.

If ξ_2 is ramified cubic, $\mathfrak{J}^{\mathfrak{s}} = \mathrm{SL}_3(\mathbb{C})$ and $\mathcal{H}^{\mathfrak{s}} = \mathcal{H}(\mathrm{PGL}_3(F), 1)$. [9, Section 9.3.1 case (a)] and [50, Table 4.1], $\pi(\nu^{\mp 1} \otimes \xi_2)$ and $J(\nu^{\mp 1} \otimes \xi_2)$ correspond to the standard module indexed by $(t_b, e_{\alpha_2}, 1)$ and $(t_b, 0, 1)$ respectively. We conclude that

$$D_{\mathcal{H}^{\mathfrak{s}}}([M_{t_b, e_{\alpha_2}, 1}]) = [M^*_{t_b, e_{\alpha_2}, 1}] = [M_{t_b, 0, 1}].$$

We can see easily that if we require $I^\alpha(\nu^{\mp 1} \otimes \xi_2\nu)$ to be irreducible, then ξ_2 is not quadratic. If ξ_2 is unramified, then $I^\alpha(\nu^{\mp 1} \otimes \xi_2\nu)$ being irreducible forces it not to be of any finite order.

If ξ_2 is ramified non-cubic, $\mathfrak{J}^{\mathfrak{s}} = \mathrm{GL}_2(\mathbb{C})$ and $\mathcal{H}^{\mathfrak{s}} = \mathcal{H}(\mathrm{GL}_2(F), 1)$. By [9, Section 9.3.1 case (b)] and [50, Table 2.1], $\pi(\nu^{\mp 1} \otimes \xi_2)$ and $J(\nu^{\mp 1} \otimes \xi_2)$ correspond to the standard module indexed by $(t_a, e_{\alpha_1}, 1)$ and $(t_a, 0, 1)$ respectively. We conclude that

$$D_{\mathcal{H}^{\mathfrak{s}}}([M_{t_a, e_{\alpha_1}, 1}]) = [M^*_{t_a, e_{\alpha_1}, 1}] = [M_{t_a, 0, 1}].$$

If ξ_2 is unramified and $s_2 \neq \pm 1$ (required by $I^\beta(\nu^{\mp 1} \otimes \xi_2\nu)$ being irreducible), then $\mathfrak{J}^{\mathfrak{s}} = G_2(\mathbb{C})$ and $\mathcal{H}^{\mathfrak{s}} = \mathcal{H}(G_2(F), 1)$. By [9, Section 9.3.1 case (c)] and [50, Table 6.1], $\pi(\nu^{\mp 1} \otimes \xi_2)$ and $J(\nu^{\mp 1} \otimes \xi_2)$ correspond to the standard module indexed by $(t_g, e_{\alpha_1}, 1)$ and $(t_g, 0, 1)$ respectively. We conclude that

$$D_{\mathcal{H}^{\mathfrak{s}}}([M_{t_g, e_{\alpha_1}, 1}]) = [M^*_{t_g, e_{\alpha_1}, 1}] = [M_{t_g, 0, 1}].$$

7.7 Two lemmas for computation

Before we discuss the case when $I^{\alpha}(\nu^{\mp 1}\otimes\xi_2\nu)$ is reducible, we need to introduce some lemmas for computational convenience.

Lemma 7.7.1. *Let $\chi\in\mathrm{Irr}(F^{\times})$. We have in $\mathrm{R}(L_{\alpha})$:*

$$r_{\alpha}\left(I_{\alpha}\left(s,\delta(\chi)\right)\right)=\nu^{s}\delta(\chi)+\nu^{-s}\delta(\chi^{-1})+I^{\alpha}(\nu^{2s}\chi^{2}\otimes\nu^{-s+1/2}\chi^{-1})+I^{\alpha}(\nu^{s+1/2}\chi\otimes\nu^{-2s}\chi^{-2}).$$
$$r_{\alpha}\left(I_{\alpha}(s,\chi\circ\det)\right)=\nu^{s}\chi\circ\det+\nu^{-s}\chi^{-1}\circ\det+I^{\alpha}(\nu^{s-1/2}\chi\otimes\nu^{-2s}\chi^{-2})+I^{\alpha}(\nu^{2s}\chi^{2}\otimes\nu^{-s-1/2}\chi^{-1}).$$
$$r_{\alpha}\left(I_{\beta}\left(s,\delta(\chi)\right)\right)=I^{\alpha}(\nu^{s-1/2}\chi\otimes\nu)+I^{\alpha}(\nu\otimes\nu^{-(s+1/2)}\chi^{-1})+I^{\alpha}(\nu^{s+1/2}\chi\otimes\nu^{-s+1/2}\chi^{-1}).$$
$$r_{\alpha}\left(I_{\beta}\left(s,\chi\circ\det\right)\right)=I^{\alpha}(\nu^{s+1/2}\chi\otimes\nu^{-1})+I^{\alpha}(\nu^{-1}\otimes\nu^{-s+1/2}\chi^{-1})+I^{\alpha}(\nu^{s-1/2}\chi\otimes\nu^{-s-1/2}\chi^{-1}).$$

If we use [15, Lemma 5.4 (iii)], we also have:

$$r_{\alpha}\left(I_{\beta}(s,\chi\circ\det)\right)=I^{\alpha}(\nu^{-s+1/2}\chi^{-1}\otimes\nu^{-1})+I^{\alpha}(\nu^{-1}\otimes\nu^{s+1/2}\chi)+I^{\alpha}(\nu^{-s-1/2}\chi^{-1}\otimes\nu^{s-1/2}\chi).$$

Lemma 7.7.2. *Let $\chi\in\mathrm{Irr}(F^{\times})$. In $\mathrm{R}(L_{\beta})$, we have:*

$$r_{\beta}\left(I_{\beta}\left(s,\delta(\chi)\right)\right)=\nu^{s}\delta(\chi)+\nu^{-s}\delta(\chi^{-1})+I^{\beta}(\nu\otimes\nu^{s-1/2}\chi)+I^{\beta}(\nu^{-s+1/2}\chi^{-1}\otimes\nu^{s+1/2}\chi).$$
$$r_{\beta}\left(I_{\beta}(s,\chi\circ\det)\right)=\nu^{s}\chi\circ\det+\nu^{-s}\chi^{-1}\circ\det+I^{\beta}(\nu^{-1}\otimes\nu^{s+1/2}\chi)+I^{\beta}(\nu^{-s-1/2}\chi^{-1}\otimes\nu^{s-1/2}\chi).$$
$$r_{\beta}\left(I_{\alpha}(s,\delta(\chi))\right)=I^{\beta}(\nu^{s+1/2}\chi\otimes\nu^{s-1/2}\chi)+I^{\beta}(\nu^{-2s}\chi^{-2}\otimes\nu^{s+1/2}\chi)+I^{\beta}(\nu^{-s+1/2}\chi^{-1}\otimes\nu^{2s}\chi^{2}).$$
$$r_{\beta}\left(I_{\alpha}(s,\chi\circ\det)\right)=I^{\beta}(\nu^{s-1/2}\chi\otimes\nu^{s+1/2}\chi)+I^{\beta}(\nu^{-2s}\chi^{-2}\otimes\nu^{s-1/2}\chi)+I^{\beta}(\nu^{-s-1/2}\chi^{-1}\otimes\nu^{2s}\chi^{2}).$$

In the following sections, we will determine how the duality operator acts when $I_{\gamma}(s,\delta(\chi))$ and $I_{\gamma}(s,\chi\circ\det)$ ($\gamma=\alpha$ or β) reduce. We will discuss based on the value of s and χ in $I_{\alpha}(s,\delta(\chi))$ and $I_{\alpha}(s,\chi\circ\det)$. Excluding the cases listed in [47, Theorem 3.1], [47, Lemma 3.1] tells us that $D_{G_2}(I_{\gamma}(s,\delta(\chi)))=I_{\gamma}(s,\chi\circ\det)$ if χ is a unitary character.

7.8 Case by case discussion, continued

7.8.1 Case (2) within the case $\mathfrak{s}=[T,1]_G$ Part I

We start by discussing the case when $I^{\alpha}(\nu^{\mp 1-s}\otimes\xi_2\nu^{-s})$ is reducible. This condition is equivalent to

$$\nu^{\mp 1}\otimes\xi_2\in\{\nu\otimes 1,\ \nu\otimes\nu^{2},\ \nu^{-1}\otimes 1,\ \nu^{-1}\otimes\nu^{-2}\}.$$

We first discuss the case when $\nu^{\mp 1}\otimes\xi_2\in\{\nu\otimes 1,\ \nu^{-1}\otimes 1\}$. This is equivalent to $s=1/2$ and $\chi=1$ in Lemma 7.7.1 and Lemma 7.7.2. This part is a special case of the cases when $s=1/2$ and $\chi^2=1$ or $\chi^3=1$, but the arguments are slightly different because

(a) Some representations will be reducible, like $I^{\alpha}(\nu\otimes 1)$ in (7.9.2).

(b) $\nu\otimes 1$ and $\nu\otimes\nu^{-1}$ are not regular *i.e.* $s_\alpha s_\beta s_\alpha(\nu\otimes 1)=\nu\otimes 1$, we can not do the length estimate.

(c) We have less isomorphic representations.

If $s=1/2$ and $\chi=1$, then by [47, Theorem 3.1] the four terms $I_\gamma(s,\delta(\chi))$ and $I_\gamma(s,\chi\circ\det)$, $\gamma=\alpha$, β all reduces.

If $s=1/2$ and $\chi=1$, then $I(1\otimes\nu^{-1})=I(\nu^{-1}\otimes\nu)=I(1\otimes\nu)$ $[\mathrm{R}(G_2)]$ (we omit the terms differ by s_α). And as representations, we have one isomorphism only $I(\nu\otimes\nu^{-1})\cong I(\nu^{-1}\otimes\nu)$.

We have

$$I_\beta(\nu^{1/2}\delta(1))+I_\beta(\nu^{1/2}\circ\det)(=I(1\otimes\nu))=I_\alpha(\nu^{1/2}\circ\det)+I_\alpha(\nu^{1/2}\delta(1))\ [\mathrm{R}(G_2)],\quad (7.8.1)$$

where $I_\beta(\nu^{1/2}\delta(1))$ and $I_\alpha(\nu^{1/2}\circ\det)$ are both subrepresentations of $I(1\otimes\nu)$.

Corollary 7.8.1. *If we take $s=1/2$, $\chi=1$ in Lemma 7.7.1, we have in $\mathrm{R}(L_\alpha)$:*

$$\begin{aligned} r_\alpha(I_\alpha(1/2,\delta(1)))&=\nu^{1/2}\delta(1)+\nu^{-1/2}\delta(1)+I^\alpha(\nu\otimes 1)+I^\alpha(\nu\otimes\nu^{-1})\\ &=2\nu^{1/2}\delta(1)+\nu^{-1/2}\delta(1)+\nu^{1/2}\circ\det+I^\alpha(\nu\otimes\nu^{-1}).\end{aligned}\quad (7.8.2)$$

$$\begin{aligned} r_\alpha(I_\alpha(1/2,1_{\mathrm{GL}_2}))&=\nu^{1/2}\circ\det+\nu^{-1/2}\circ\det+I^\alpha(1\otimes\nu^{-1})+I^\alpha(\nu\otimes\nu^{-1})\\ &=\nu^{1/2}\circ\det+2\nu^{-1/2}\circ\det+\nu^{-1/2}\delta(1)+I^\alpha(\nu\otimes\nu^{-1}).\end{aligned}\quad (7.8.3)$$

$$\begin{aligned} r_\alpha(I_\beta(1/2,\delta(1)))&=I^\alpha(1\otimes\nu)+I^\alpha(\nu\otimes\nu^{-1})+I^\alpha(\nu\otimes 1)\\ &=2\nu^{1/2}\delta(1)+2\nu^{1/2}\circ\det+I^\alpha(\nu\otimes\nu^{-1}).\end{aligned}\quad (7.8.4)$$

$$\begin{aligned} r_\alpha(I_\beta(1/2,1_{\mathrm{GL}_2}))&=I^\alpha(\nu\otimes\nu^{-1})+I^\alpha(\nu^{-1}\otimes 1)+I^\alpha(1\otimes\nu^{-1})\\ &=2\nu^{-1/2}\delta(1)+2\nu^{-1/2}\circ\det+I^\alpha(\nu\otimes\nu^{-1}).\end{aligned}\quad (7.8.5)$$

Corollary 7.8.2. *If we take $s=1/2$, $\chi=1$ in lemma 7.7.2, we have in $\mathrm{R}(L_\beta)$:*

$$\begin{aligned} r_\beta(I_\beta(1/2,\delta(1)))&=\nu^{1/2}\delta(1)+\nu^{-1/2}\delta(1)+I^\beta(\nu\otimes 1)+I^\beta(1\otimes\nu)\\ &=2\nu^{1/2}\delta(1)+\nu^{-1/2}\delta(1)+\nu^{1/2}\circ\det+I^\beta(\nu\otimes 1).\end{aligned}\quad (7.8.6)$$

$$\begin{aligned} r_\beta(I_\beta(1/2,1_{\mathrm{GL}_2}))&=\nu^{1/2}\circ\det+\nu^{-1/2}\circ\det+I^\beta(\nu^{-1}\otimes\nu)+I^\beta(\nu^{-1}\otimes 1)\\ &=\nu^{1/2}\circ\det+2\nu^{-1/2}\circ\det+\nu^{-1/2}\delta(1)+I^\beta(\nu^{-1}\otimes 1).\end{aligned}\quad (7.8.7)$$

$$\begin{aligned} r_\beta(I_\alpha(1/2,\delta(1)))&=I^\beta(\nu\otimes 1)+I^\beta(\nu^{-1}\otimes\nu)+I^\beta(1\otimes\nu)\\ &=\nu^{1/2}\delta(1)+\nu^{1/2}\circ\det+\nu^{-1/2}\delta(1)+\nu^{-1/2}\circ\det+I^\beta(\nu\otimes 1).\end{aligned}\quad (7.8.8)$$

$$\begin{aligned} r_\beta\left(I_\alpha(1/2, 1_{\mathrm{GL}_2})\right) &= I^\beta(1\otimes\nu) + I^\beta(\nu^{-1}\otimes 1) + I^\beta(\nu^{-1}\otimes\nu) \\ &= \nu^{-1/2}\delta(1) + \nu^{-1/2}\circ\det + \nu^{1/2}\delta(1) + \nu^{1/2}\circ\det + I^\beta(\nu^{-1}\otimes 1). \end{aligned} \tag{7.8.9}$$

In (7.8.9) and (7.8.8), we used [15, Lemma 5.4 (iii)]:

$$I^\beta(\chi_1\otimes\chi_2) = I^\beta\left(s_\beta(\chi_1\otimes\chi_2)\right).$$

in $\mathrm{R}(L_\beta)$.

The following Proposition is from [47], but our proof is a bit different which enables us to know more about the images of the composition factors under Jacquet functors.

Proposition 7.8.3 (G. Muic Proposition 4.3)**.** *Suppose that* $\chi = 1$, $s = 1/2$ *then we have the following.*

(1) The induced representation $I(1\otimes\nu)$ *contains exactly two irreducible subrepresentations* $\pi(1)$ *and* $\pi'(1)$*. We have* $r_T^{G_2}(\pi(1)) = 1\otimes\nu$, $r_T^{G_2}(\pi'(1)) = 1\otimes\nu + 2(\nu\otimes 1)$.

(2) In $\mathrm{R}(G_2)$*, we have*

$$\begin{aligned} I_\alpha(1/2,\delta(1)) &= \pi'(1) + J_\alpha(1/2,\delta(1)) + J_\beta(1/2,\delta(1)), \\ I_\beta(1/2,\delta(1)) &= \pi(1) + \pi'(1) + J_\beta(1/2,\delta(1)), \\ I_\alpha(1/2, 1_{\mathrm{GL}_2}) &= \pi(1) + J_\beta(1,\pi(1,1)) + J_\beta(1/2,\delta(1)), \\ I_\beta(1/2, 1_{\mathrm{GL}_2}) &= J_\beta(1,\pi(1,1)) + J_\beta(1/2,\delta(1)) + J_\alpha(1/2,\delta(1)). \end{aligned} \tag{7.8.10}$$

Proof. By Langlands quotient theorem applied to the following: $I(\nu\otimes 1) = I_\beta(1,\pi(1,1))$, we see $J_\beta(1,\pi(1,1))$ is the unique irreducible quotient of $I(\nu\otimes 1)$. Consider the exact sequence:

$$0 \to I_\alpha(1/2,\delta(1)) \to I(\nu\otimes 1) \to I_\alpha(1/2, 1_{\mathrm{GL}_2}) \to 0.$$

We know $J_\beta(1,\pi(1,1))$ is the unique irreducible quotient of $I_\alpha(1/2, 1_{\mathrm{GL}_2})$.

Then Langlands quotient theorem applied to $I_\alpha(1/2,\delta(1))$ and $I_\beta(1/2,\delta(1))$, we find two more composition factors: $J_\alpha(1/2,\delta(1))$ and $J_\beta(1/2,\delta(1))$ as the unique irreducible quotient of $I_\alpha(1/2,\delta(1))$ and $I_\beta(1/2,\delta(1))$ respectively.

We have

$$J_\beta(1/2,\delta(1)) \hookrightarrow I_\beta(-1/2,\delta(1)) \hookrightarrow I(\nu^{-1}\otimes\nu) \cong I(\nu\otimes\nu^{-1}). \tag{7.8.11}$$

From

$$0 \neq \mathrm{Hom}_{G_2}\left(J_\beta\left(1/2, \delta(1)\right), I(\nu^{-1} \otimes \nu)\right) = \mathrm{Hom}_{L_\alpha}\left(r_\alpha\left(J_\beta(1/2, \delta(1))\right), I^\alpha(\nu^{-1} \otimes \nu)\right),$$
$$0 \neq \mathrm{Hom}_{G_2}\left(J_\beta\left(1/2, \delta(1)\right), I_\beta\left(-1/2, \delta(1)\right)\right) = \mathrm{Hom}_{L_\beta}\left(r_\beta\left(J_\beta\left(1/2, \delta(1)\right)\right), \nu^{-1/2}\delta(1)\right)\Big),$$
$$0 \neq \mathrm{Hom}_{G_2}\left(J_\beta(1/2, \delta(1)), I(\nu^{-1} \otimes \nu)\right) = \mathrm{Hom}_{L_\beta}\left(r_\beta\left(J_\beta\left(1/2, \delta(1)\right)\right), I^\beta(\nu \otimes \nu^{-1})\right).$$

We conclude that $r_\alpha(J_\beta(1/2, \delta(1)))$ contains $I^\alpha(\nu^{-1} \otimes \nu)$, $r_\beta(J_\beta(1/2, \delta(1)))$ contains $\nu^{-1/2}\delta(1)$ and $r_\beta(J_\beta(1/2, \delta(1)))$ also contains $\nu^{1/2}\delta(1)$ or $\nu^{1/2} \circ \det$, or both.

We know that $I_\beta(1/2, 1_{\mathrm{GL}_2})$ is a subrepresentation of $I(\nu \otimes \nu^{-1})$ and we must have

$$I_\beta\left(1/2, \delta(1)\right)\left(= I_\beta\left(-1/2, \delta(1)\right)[\mathrm{R}(G_2)]\right) \bigcap_{[\mathrm{R}(G_2)]} I_\beta(1/2, 1_{\mathrm{GL}_2}) \neq 0, \tag{7.8.12}$$

(where the intersection $\bigcap_{[\mathrm{R}(G_2)]}$ in the above means we take their common factors in $[\mathrm{R}(G_2)]$), otherwise

$$J_\beta(1/2, \delta(1)) \hookrightarrow I_\beta(-1/2, \delta(1)) \hookrightarrow I(\nu \otimes \nu^{-1})/I_\beta(1/2, 1_{\mathrm{GL}_2}) \cong I_\beta(1/2, \delta(1)).$$

This implies that $J_\beta(1/2, \delta(1))$ is a subrepresentation of $I_\beta\left(1/2, \delta(1)\right)$, then $I_\beta(1/2, \delta(1))$ is irreducible. This contradicts [47, Theorem 3.1].

From (7.8.12), we compare (7.8.4) and (7.8.5) to find out the image of common factor under r_α is $I^\alpha(\nu \otimes \nu^{-1})$, thus the only common factor of $I_\beta(1/2, \delta(1)) \bigcap_{[\mathrm{R}(G_2)]} I_\beta(1/2, 1_{\mathrm{GL}_2})$ has to be $J_\beta(1/2, \delta(1))$, and $r_\alpha(J_\beta(1/2, \delta(1))) = I^\alpha(\nu \otimes \nu^{-1})$, $r_\beta(J_\beta(1/2, \delta(1))) = \nu^{-1/2}\delta(1) + \nu^{1/2} \circ \det\ [\mathrm{R}(L_\beta)]$. We find from (7.8.2) and (7.8.3) that $I_\alpha(1/2, 1_{\mathrm{GL}_2})$ and $I_\alpha(1/2, \delta(1))$ both contain $J_\beta(1/2, \delta(1))$ as a composition factor.

Recall that $I_\beta(1/2, \delta(1))$, $I_\alpha(1/2, 1_{\mathrm{GL}_2})$ are both subrepresentations of $I(1 \otimes \nu)$. We apply $-2J_\beta(1/2, \delta(1))$ to both sides of (7.8.1), and get

$$\begin{aligned} &I_\beta(1/2, \delta(1)) - J_\beta(1/2, \delta(1)) + I_\beta(1/2, 1_{\mathrm{GL}_2}) - J_\beta(1/2, \delta(1)) \\ =&I_\alpha(1/2, 1_{\mathrm{GL}_2}) - J_\beta(1/2, \delta(1)) + I_\alpha(1/2, \delta(1)) - J_\beta(1/2, \delta(1))\ [\mathrm{R}(G_2)], \end{aligned} \tag{7.8.13}$$

where $(I_\beta(1/2, \delta(1)) - J_\beta(1/2, \delta(1))) \bigcap_{[\mathrm{R}(G_2)]} (I_\beta(1/2, 1_{\mathrm{GL}_2}) - J_\beta(1/2, \delta(1))) = \varnothing\ [\mathrm{R}(G_2)]$ by comparing the common terms in (7.8.4) and (7.8.5). We have

$$\left(I_\beta(1/2, \delta(1)) \bigcap_{[\mathrm{R}(G_2)]} I_\alpha(1/2, 1_{\mathrm{GL}_2})\right) - J_\beta(1/2, \delta(1)) \neq 0.$$

Otherwise

$$I_\alpha(1/2, 1_{\mathrm{GL}_2}) - J_\beta(1/2, \delta(1)) \subset I_\beta(1/2, 1_{\mathrm{GL}_2}) - J_\beta(1/2, \delta(1))\ [\mathrm{R}(G_2)]$$

can be deduced from (7.8.13) whose left hand side is a disjoint sum, but this will contradict the fact that excluding the terms brought by $J_\beta(1/2, \delta(1))$, the remaining of (7.8.3) is not included in the remaining of (7.8.5). Compare (7.8.3) and (7.8.4) (for r_β, (7.8.9) and (7.8.6) respectively), we know that besides $J_\beta(1/2, \delta(1))$, there is another irreducible subquotient contained in $I_\beta(1/2, \delta(1)) \bigcap_{[\mathrm{R}(G_2)]} I_\alpha(1/2, 1_{\mathrm{GL}_2})$ whose image under r_α is $\nu^{1/2}\det$ and under r_β is $\nu^{1/2}\delta(1)$. If we look at any irreducible subrepresentation of $I_\alpha(\nu^{1/2} \circ \det)$, by the adjointness of r_α and I_α, we know the image of such irreducible subrepresentation under r_α must contain $\nu^{1/2}\det$. From the fact that $r_\alpha(I_\alpha(1/2, 1_{\mathrm{GL}_2}))$ contains $\nu^{1/2}\det$ with multiplicity one, we know the other composition factor of $I_\beta(1/2, \delta(1)) \bigcap_{[\mathrm{R}(G_2)]} I_\alpha(1/2, 1_{\mathrm{GL}_2})$ is the unique irreducible subrepresentation of $I_\alpha(1/2, 1_{\mathrm{GL}_2})$, denoted by $\pi(1)$.

Using the same argument to (7.8.2) and (7.8.5) (for r_β, (7.8.7) and (7.8.8), respectively), we know that there is an irreducible subquotient of

$$\left(I_\beta(1/2, 1_{\mathrm{GL}_2}) \bigcap_{[\mathrm{R}(G_2)]} I_\alpha(1/2, \delta(1)) \right) - J_\beta(1/2, \delta(1))$$

whose image under r_α is $\nu^{-1/2}\delta(1)$, under r_β is $\nu^{-1/2} \circ \det$. But we have

$$J_\alpha(1/2, \delta(1)) \hookrightarrow I_\alpha(-1/2, \delta(1)) \, (= I_\alpha \, (1/2, \delta(1)) \, [\mathrm{R}(G_2)]) \tag{7.8.14}$$

as the unique subrepresentation, $r_\alpha(J_\alpha(1/2, \delta(1)))$ contains $\nu^{-1/2}\delta(1)$. The multiplicity of $\nu^{-1/2}\delta(1)$ in (7.8.2) is one, we thus conclude

$$I_\beta(1/2, 1_{\mathrm{GL}_2}) \bigcap_{[\mathrm{R}(G_2)]} I_\alpha(1/2, \delta(1)) = J_\beta(1/2, \delta(1)) + J_\alpha(1/2, \delta(1)).$$

From the inclusion

$$J_\beta(1, \pi(1,1)) \hookrightarrow I_\alpha(-1/2, 1_{\mathrm{GL}_2}),$$

we know $r_\alpha(J_\beta(1, \pi(1,1)))$ contains $\nu^{-1/2} \circ \det$. Since $I_\alpha(-1/2, 1_{\mathrm{GL}_2}) \hookrightarrow I(\nu^{-1} \otimes 1)$, we know $r_\beta(J_\beta(1, \pi(1,1)))$ contains $I^\beta(\nu^{-1} \otimes 1)$.

We may apply similar argument to any irreducible subrepresentation of $I_\alpha(1/2, \delta(1))$, and we find the image under r_α must contain $\nu^{1/2}\delta(1)$, the image under r_β of such irreducible subrepresentation must contain $I^\beta(\nu \otimes 1)$ which is of multiplicity 1 in $r_\beta(I_\alpha(1/2, \delta(1)))$, thus there is a unique irreducible subrepresentation of $I_\alpha(1/2, \delta(1))$. We denote this representation by $\pi'(1)$.

Suppose that $r_\beta \pi'(1) = I^\beta(\nu \otimes 1) + m\nu^{1/2}\delta(1)$ where $m = 0$ or 1. If $m = 0$, then there may exist another subquotient τ such that $r_\beta(\tau) = \nu^{1/2}\delta(1)$, if we assume

$$r_\alpha(\pi'(1)) = \nu^{1/2}\delta(1) + s\nu^{1/2}\delta(1) + t\nu^{1/2} \circ \det$$

where $s, t = 0$ or 1 but they do not equal to 1 at the same time, then

$$r_\alpha(\tau) = (1-s)\nu^{1/2}\delta(1) + (1-t)\nu^{1/2} \circ \det .$$

If $s = 0$, $t = 1$, then

$$0 \neq \mathrm{Hom}_{L_\alpha}(r_\alpha(\tau), \nu^{1/2}\delta(1)) = \mathrm{Hom}_{G_2}(\tau, I_\alpha(1/2, \delta(1)))$$

would contradicts the uniqueness of $\pi'(1)$ as subrepresentation of $I_\alpha(1/2, \delta(1))$. Similarly, if $s = 1$, $t = 0$, then it leads to a contradiction to uniqueness of $\pi(1)$. If $s = t = 0$, then $r_\alpha D_{G_2}(\pi'(1))$ is not included in $r_\alpha(I_\alpha(1/2, 1_{\mathrm{GL}_2}))$ contradicts [47, Lemma 3.1].

We conclude that $m = 1$, and

$$r_\beta\pi'(1) = I^\beta(\nu \otimes 1) + \nu^{1/2}\delta(1), \ r_\alpha\pi'(1) = 2\nu^{1/2}\delta(1) + \nu^{1/2} \circ \det .$$

By the same argument,

$$r_\beta(J_\beta(1, \pi(1,1))) = I^\beta(\nu^{-1} \otimes 1) + \nu^{-1/2} \circ \det,$$

and

$$r_\alpha(J_\beta(1, \pi(1,1))) = 2\nu^{-1/2} \det + \nu^{-1/2}\delta(1).$$

□

We list the images of composition factors under r_α and r_β in the following proposition:

Proposition 7.8.4. *Based on the proof of Proposition 7.8.3, we have the following*

$$\begin{gathered} r_\alpha(\pi(1)) = \nu^{1/2} \circ \det, \ r_\beta(\pi(1)) = \nu^{1/2}\delta(1), \\ r_\alpha(J_\alpha(1/2, \delta(\chi))) = \nu^{-1/2}\delta(1), \ r_\beta(J_\alpha(1/2, \delta(\chi))) = \nu^{-1/2} \circ \det, \\ r_\alpha\left(J_\beta(1/2, \delta(1))\right) = I^\alpha(\nu \otimes \nu^{-1}), \ r_\beta\left(J_\beta(1/2, \delta(1))\right) = \nu^{-1/2}\delta(1) + \nu^{1/2} \circ \det, \\ r_\alpha(\pi'(1)) = 2\nu^{1/2}\delta(1) + \nu^{1/2} \circ \det, \ r_\beta(\pi'(1)) = I^\beta(\nu \otimes 1) + \nu^{1/2}\delta(1), \\ r_\alpha(J_\beta(1, \pi(1,1))) = 2\nu^{-1/2} \circ \det + \nu^{-1/2}\delta(1), \ r_\beta(J_\beta(1, \pi(1,1))) = I^\beta(\nu^{-1} \otimes 1) + \nu^{-1/2} \circ \det . \end{gathered} \tag{7.8.15}$$

Proposition 7.8.5. *We compute the Aubert-Zelevinsky duality of all the irreducible representations listed above:*

$$\begin{gathered} D_{G_2}(\pi(1)) = J_\alpha(1/2, \delta(1)), \quad D_{G_2}(J_\beta(1/2, \delta(1))) = J_\beta(1/2, \delta(1)) \\ D_{G_2}(\pi'(1)) = J_\beta(1, \pi(1,1)). \end{gathered} \tag{7.8.16}$$

Proof. We compute as follows

$$\begin{aligned}
D_{G_2}(\pi(1)) &= I\circ r_T^{G_2}(\pi(1)) - I_\alpha\circ r_\alpha(\pi(1)) - I_\beta\circ r_\beta(\pi(1)) + \pi(1)\\
&= I\circ(r_T^{L_\alpha}(\nu^{1/2}\circ\det)) - I_\alpha(\nu^{1/2}\circ\det) - I_\beta(\nu^{1/2}\delta(1)) + \pi(1)\\
&= I_\alpha(1/2,\delta(1)) - I_\beta(1/2,\delta(1)) + \pi(1)\\
&= J_\alpha(1/2,\delta(1)).
\end{aligned} \tag{7.8.17}$$

Similarly for the rest terms:

$$\begin{aligned}
&D_{G_2}(J_\beta(1/2,\delta(1))) = I\circ r_T^{G_2}(J_\beta(1/2,\delta(1))) - I_\alpha\circ r_\alpha(J_\beta(1/2,\delta(1)))\\
&- I_\beta\circ r_\beta(J_\beta(1/2,\delta(1))) + J_\beta(1/2,\delta(1))\\
&= I\circ(r_T^{L_\alpha}(I^\alpha(\nu\otimes\nu^{-1}))) - I_\alpha(I^\alpha(\nu\otimes\nu^{-1})) - I_\beta(\nu^{-1/2}\delta(1))\\
&- I_\beta(\nu^{1/2}\circ\det) + J_\beta(1/2,\delta(1))\\
&= J_\beta(1/2,\delta(1)).
\end{aligned} \tag{7.8.18}$$

$$\begin{aligned}
&D_{G_2}(\pi'(1)) = I\circ r_T^{G_2}(\pi'(1)) - I_\alpha\circ r_\alpha(\pi'(1)) - I_\beta\circ r_\beta(\pi'(1)) + \pi'(1)\\
&= I\circ(r_T^{L_\alpha}(2\nu^{1/2}\delta(1) + \nu^{1/2}\circ\det)) - I_\alpha(2\nu^{1/2}\delta(1) + \nu^{1/2}\circ\det)\\
&- I_\beta(I^\beta(\nu\otimes 1) + \nu^{1/2}\delta(1)) + \pi'(1)\\
&= 2I(\nu\otimes 1) - 2I_\alpha(1/2,\delta(1)) - I_\alpha(1/2,1_{\mathrm{GL}_2}) - I_\beta(1/2,\delta(1)) + \pi'(1)\\
&= I_\alpha(1/2,1_{\mathrm{GL}_2}) - I_\beta(1/2,\delta(1)) + \pi'(1)\\
&= J_\beta(1,\pi(1,1)).
\end{aligned} \tag{7.8.19}$$

□

In this case $\mathfrak{s} = [T,1]$ and $\mathfrak{J}^{\mathfrak{s}} = G_2(\mathbb{C})$ and $\mathcal{H}^{\mathfrak{s}} = \mathcal{H}(G_2(F),1)$. By [9, Section 9.3.1 table 15] and [50, Table 6.1], we have the following proposition.

Proposition 7.8.6. *Using the indexing triples to denote the standard modules, we have:*

$$\begin{aligned}
D_{\mathcal{H}^{\mathfrak{s}}}(M_{t_e,e_{\alpha^\vee}+e_{\alpha^\vee}+2\beta^\vee,(21)}) &= M^*_{t_e,e_{\alpha^\vee}+e_{\alpha^\vee}+2\beta^\vee,(21)} = M_{t_e,e_{\alpha^\vee},1},\\
D_{\mathcal{H}^{\mathfrak{s}}}(M_{t_e,e_{\alpha^\vee+\beta^\vee},1}) &= M^*_{t_e,e_{\alpha^\vee+\beta^\vee},1} = M_{t_e,e_{\alpha^\vee+\beta^\vee},1}\\
D_{\mathcal{H}^{\mathfrak{s}}}(M_{t_e,e_{\alpha^\vee}+e_{\alpha^\vee}+2\beta^\vee,(3)}) &= M^*_{t_e,e_{\alpha^\vee}+e_{\alpha^\vee}+2\beta^\vee,(3)} = M_{t_e,0,1}.
\end{aligned} \tag{7.8.20}$$

In the next subsection, we continue to discuss the other possibility for $I^\alpha(\nu^{\mp 1-s}\otimes\xi_2\nu^{-s})$ to be reducible.

7.8.2 Case (2) within the case $\mathfrak{s} = [T,1]_G$ Part II

$\nu^{\mp 1}\otimes\xi_2 \in \{\nu\otimes\nu^2,\ \nu^{-1}\otimes\nu^{-2}\}$ This reduces to the case $s = 3/2$ and $\chi = 1$ in Lemma 7.7.1 and Lemma 7.7.2. If $s = 3/2$ and $\chi = 1$, then by [47, Theorem 3.1], $I_\alpha(s,\delta(\chi))$ and $I_\alpha(s,\chi\circ\det)$ reduce, while $I_\beta(s,\delta(\chi))$ or $I_\beta(s,\chi\circ\det)$ do not.

If $s = 3/2$ and $\chi = 1$, then $I(\nu^2 \otimes \nu) = I(\nu^3 \otimes \nu^{-2}) = I(\nu \otimes \nu^{-3}) = I(\nu^{-2} \otimes \nu^{-1}) = I(\nu^3 \otimes \nu^{-1}) = I(\nu^2 \otimes \nu^{-3})$ [R(G_2)] (we omit the terms differ by s_α). Therefore $I_\beta(\nu^{5/2} \circ \det) + I_\beta(\nu^{5/2}\delta(1)) = I(\nu^2 \otimes \nu) = I_\alpha(\nu^{3/2} \circ \det) + I_\alpha(\nu^{3/2}\delta(1))$ [R(G_2)].

Corollary 7.8.7. *We consider the case when $s = 3/2$ and $\chi = 1$ in Lemma 7.7.1, list the equations below:*

$$r_\alpha(I_\alpha(3/2, \delta(1))) = \nu^{3/2}\delta(1) + \nu^{-3/2}\delta(1) + I^\alpha(\nu^3 \otimes \nu^{-1}) + I^\alpha(\nu^2 \otimes \nu^{-3}), \tag{7.8.21}$$

$$r_\alpha(I_\alpha(3/2, 1_{\mathrm{GL}_2})) = \nu^{3/2} \circ \det + \nu^{-3/2} \circ \det + I^\alpha(\nu \otimes \nu^{-3}) + I^\alpha(\nu^3 \otimes \nu^{-2}), \tag{7.8.22}$$

$$r_\alpha(I_\beta(5/2, \delta(1))) = \nu^{3/2}\delta(1) + \nu^{3/2} \circ \det + I^\alpha(\nu \otimes \nu^{-3}) + I^\alpha(\nu^3 \otimes \nu^{-2}), \tag{7.8.23}$$

$$r_\alpha(I_\beta(5/2, 1_{\mathrm{GL}_2})) = \nu^{-3/2}\delta(1) + \nu^{-3/2} \circ \det + I^\alpha(\nu^3 \otimes \nu^{-1}) + I^\alpha(\nu^2 \otimes \nu^{-3}). \tag{7.8.24}$$

Corollary 7.8.8. *We consider the case when $s = 3/2$ and $\chi = 1$ in Lemma 7.7.2, list the equations below:*

$$r_\beta(I_\beta(5/2, \delta(1))) = \nu^{5/2}\delta(1) + \nu^{-5/2}\delta(1) + I^\beta(\nu \otimes \nu^2) + I^\beta(\nu^{-2} \otimes \nu^3), \tag{7.8.25}$$

$$r_\beta(I_\beta(5/2, 1_{\mathrm{GL}_2})) = \nu^{5/2} \circ \det + \nu^{-5/2} \circ \det + I^\beta(\nu^{-1} \otimes \nu^3) + I^\beta(\nu^{-3} \otimes \nu^2), \tag{7.8.26}$$

$$r_\beta(I_\alpha(3/2, \delta(1))) = \nu^{5/2}\delta(1) + \nu^{5/2} \circ \det + I^\beta(\nu^{-3} \otimes \nu^2) + I^\beta(\nu^{-1} \otimes \nu^3), \tag{7.8.27}$$

$$r_\beta\,(I_\alpha(3/2, 1_{\mathrm{GL}_2})) = \nu^{-5/2}\delta(1) + \nu^{-5/2} \circ \det + I^\beta(\nu \otimes \nu^2) + I^\beta(\nu^{-2} \otimes \nu^3). \tag{7.8.28}$$

Proposition 7.8.9. *(1) The induced representation $I(\nu^2 \otimes \nu) = \mathrm{Ind}_B^{G_2}(\delta_{G_2}^{1/2})$, it contains a unique irreducible subrepresentation St_{G_2} (the* Steinberg representation *of G_2) and a unique irreducible quotient 1_{G_2} (the* trivial representation*). Moreover, we have $r_T^{G_2}(\mathrm{St}) = \nu^2 \otimes \nu$.*

(2) We have the following equations in $R[G_2]$:

$$\begin{aligned}
I_\alpha(3/2, \delta(1)) &= \mathrm{St}_{G_2} + J_\alpha(3/2, \delta(1)) \\
I_\beta(5/2, \delta(1)) &= \mathrm{St}_{G_2} + J_\beta(5/2, \delta(1)) \\
I_\alpha(3/2, 1_{\mathrm{GL}_2}) &= 1_{G_2} + J_\beta(5/2, \delta(1)) \\
I_\beta(5/2, 1_{\mathrm{GL}_2}) &= 1_{G_2} + J_\alpha(3/2, \delta(1))
\end{aligned} \tag{7.8.29}$$

Proof. By a theorem of F. Rodier [54], we know $I(\nu^2 \otimes \nu)$ has length 4 and contains a unique subrepresentation, more precisely $I(\nu^2 \otimes \nu) = \mathrm{Ind}_B^{G_2}(\delta_{G_2}^{1/2})$, the unique subrepresentation is St_{G_2} and it has a unique irreducible quotient 1_{G_2}.

By Langlands quotient theorem applied to $I_\alpha(3/2, \delta(\chi))$ and $I_\beta(5/2, \delta(\chi))$, we find the rest two (it has length 4) composition factors: $J_\alpha(3/2, \delta(1))$ and $J_\beta(5/2, \delta(1))$.

Since $I_\alpha(3/2, \delta(1))$ and $I_\beta(5/2, \delta(1))$ are subrepresentations of $I(\nu^2 \otimes \nu)$, they must contain St_{G_2}. And $I_\alpha(3/2, 1_{\mathrm{GL}_2})$ and $I_\beta(5/2, 1_{\mathrm{GL}_2})$ are quotients of $I(\nu^2 \otimes \nu)$, thus they both contain 1_{G_2}. Part (2) of the proposition is proved.

We know that $r_\alpha(\mathrm{St}_{G_2})$ must be contained in the common parts of (7.8.21) and (7.8.23). Since $0 \neq \mathrm{Hom}_{G_2}(\mathrm{St}_{G_2}, I_\alpha(\nu^{3/2}\delta(1))) = \mathrm{Hom}_{L_\alpha}(r_\alpha(\mathrm{St}_{G_2}), \nu^{3/2}\delta(1))$, $r_\alpha(\mathrm{St}_{G_2})$ must contain $\nu^{3/2}\delta(1)$. We easily have $r_\alpha(\mathrm{St}_{G_2}) = \nu^{3/2}\delta(1)$. Composing with $r_T^{L_\alpha}$, we prove part (1) of the proposition. □

Proposition 7.8.10. *We list the images under r_α and r_β of each composition factors:*

$$\begin{aligned}
r_\alpha(\mathrm{St}_{G_2}) &= \nu^{3/2}\delta(1), \quad r_\beta(\mathrm{St}_{G_2}) = \nu^{5/2}\delta(1),\\
r_\alpha(J_\alpha(3/2, \delta(1))) &= \nu^{-3/2}\delta(1) + I^\alpha(\nu^3 \otimes \nu^{-1}) + I^\alpha(\nu^2 \otimes \nu^{-3}),\\
r_\beta(J_\alpha(3/2, \delta(1))) &= \nu^{5/2} \circ \det + I^\beta(\nu^{-3} \otimes \nu^2) + I^\beta(\nu^{-1} \otimes \nu^3),\\
r_\alpha(1_{G_2}) &= \nu^{-3/2} \circ \det, \quad r_\beta(1_{G_2}) = \nu^{-5/2} \circ \det,\\
r_\alpha(J_\beta(5/2, \delta(1))) &= \nu^{3/2} \circ \det + I^\alpha(\nu \otimes \nu^{-3}) + I^\alpha(\nu^3 \otimes \nu^{-2}),\\
r_\beta(J_\beta(5/2, \delta(1))) &= \nu^{-5/2}\delta(1) + I^\beta(\nu \otimes \nu^2) + I^\beta(\nu^{-2} \otimes \nu^3).
\end{aligned} \tag{7.8.30}$$

Proposition 7.8.11. *We compute the Aubert-Zelevinsky duality of all the irreducible representations listed above:*

$$D_{G_2}(\mathrm{St}_{G_2}) = 1_{G_2}, \quad D_{G_2}(J_\alpha(3/2, \delta(1))) = J_\beta(5/2, \delta(1)) \tag{7.8.31}$$

Proof. This is well known. But as a test of previous proposition, we do the calculation:

$$\begin{aligned}
D_{G_2}(\mathrm{St}_{G_2}) &= I \circ r_T^{G_2}(\mathrm{St}_{G_2}) - I_\alpha \circ r_\alpha(\mathrm{St}_{G_2}) - I_\beta \circ r_\beta(\mathrm{St}_{G_2}) + \mathrm{St}_{G_2}\\
&= I(\nu^2 \otimes \nu) - I_\alpha(\nu^{3/2}\delta(1)) - I_\beta(\nu^{5/2}\delta(1)) + \mathrm{St}_{G_2}\\
&= I_\alpha(3/2, 1_{\mathrm{GL}_2}) - I_\beta(\nu^{5/2}\delta(1)) + \mathrm{St}_{G_2}\\
&= 1_{G_2}.
\end{aligned} \tag{7.8.32}$$

By [47, Lemma 3.1], we have the rest of this proposition. □

In this case $\mathfrak{s} = [T, \xi \otimes \xi]$ and $\mathfrak{J}^{\mathfrak{s}} = G_2(\mathbb{C})$ and $\mathcal{H}^{\mathfrak{s}} = \mathcal{H}(G_2(F), 1)$. By [9, Section 9.3.2 table 18] and [50, Table 6.1], we get the proposition for modules of Hecke algebra:

Proposition 7.8.12. *Using the indexing triples to denote the standard modules, we have:*

$$D_{\mathcal{H}^{\mathfrak{s}}}(M_{t_a, e_{\alpha^\vee}+e_{\beta^\vee}, 1}) = M^*_{t_a, e_{\alpha^\vee}+e_{\beta^\vee}, 1} = M_{t_a, 0, 1}, \quad D_{\mathcal{H}^{\mathfrak{s}}}(M_{t_a, e_{\alpha^\vee}, 1}) = M^*_{t_a, e_{\alpha^\vee}, 1} = M_{t_a, e_{\beta^\vee}, 1}. \tag{7.8.33}$$

We now begin to discuss case (3).

7.8.3 Case (3) and (5)

In case (3), $\xi_1 = \nu^{s_1}\xi = \nu^{s_1-s_2}\xi_2 = \nu^{\pm 1}\xi_2$. By Proposition 7.3.1, we have

$$\begin{aligned} I(\xi_1 \otimes \xi_2) &= I_\alpha\left(\delta(\nu^{\pm 1/2}\xi_2)\right) + I_\alpha\left(\nu^{\pm 1/2}\xi_2 \circ \det\right) \\ &= I_\alpha\left(s, \delta\left(\nu^{s_2-s\pm 1/2}\xi\right)\right) + I_\alpha\left(s, \nu^{s_2-s\pm 1/2}\xi \circ \det\right). \end{aligned}$$

In case (5), $\xi_1 = \nu^{s_1}\xi = \xi_2^{-2}\nu^{\pm 1}$. Thus

$$I(\xi_1 \otimes \xi_2) = I\left(\nu^{\pm 1}\xi_2^{-2} \otimes \xi_2\right) = I\left(\nu^{\pm 1}\xi_2 \otimes \xi_2\right).$$

Case (5) is reduced to case (3) above.

By [47, Theorem 3.1 (i)], if the character $\chi := \nu^{s_2-s\pm 1/2}\xi$ is unitary, then $I_\alpha(s, \delta(\chi))$ and $I_\alpha(s, \chi \circ \det)$ are irreducible unless $s = \pm 1/2, \chi$ is quadratic, $s = \pm 3/2, \chi = 1$ or $s = \pm 1/2, \chi$ is cubic. ($s = \pm 3/2, \chi = 1$ is already discussed)

7.8.4 Case (3) within the case $\mathfrak{s} = [T, \xi \otimes 1]_G$ the length 2 case

If ξ_2 is ramified, then by the discussion in [9, Section 9.3.3] ξ is not quadratic nor cubic, we have $\mathfrak{J}^{\mathfrak{s}} = \mathrm{GL}_2(\mathbb{C})$ and $\mathcal{H}^{\mathfrak{s}} = \mathcal{H}(\mathrm{GL}_2(F), 1)$. The two irreducible composition factors $I_\alpha\left(\delta(\nu^{\pm 1/2}\xi_2)\right)$ and $I_\alpha\left(\nu^{\pm 1/2}\xi_2 \circ \det\right)$ correspond to the standard module indexed by $(t_a, e_\alpha, 1)$ and $(t_a, 0, 1)$ respectively. Using the indexing triples to denote the standard modules, we conclude that

$$D_{\mathcal{H}^{\mathfrak{s}}}([M_{t_a, e_\alpha, 1}]) = [M^*_{t_a, e_\alpha, 1}] = [M_{t_a, 0, 1}].$$

7.8.5 Case (3) within the case $\mathfrak{s} = [T, 1]_G$ the length 2 case

If ξ_2 is unramified, then ξ_1 is also unramified and we are in the case $\mathfrak{s} = [T, 1]$ and $\mathfrak{J}^{\mathfrak{s}} = G_2(\mathbb{C})$ and $\mathcal{H}^{\mathfrak{s}} = \mathcal{H}(G_2(F), 1)$. By [9, Section 9.3.3 table 21] and [50, Table 6.1], $I_\alpha\left(\delta(\nu^{\pm 1/2}\xi_2)\right)$ and $I_\alpha\left(\nu^{\pm 1/2}\xi_2 \circ \det\right)$ correspond to the standard module indexed by $(t_g, e_\alpha, 1)$ and $(t_g, 0, 1)$ respectively. Using the indexing triples to denote the standard modules, we have:

$$D_{\mathcal{H}^{\mathfrak{s}}}([M_{t_g, e_\alpha, 1}]) = [M^*_{t_g, e_\alpha, 1}] = [M_{t_g, 0, 1}].$$

In the following, we will discuss the cases when

$$I_\alpha\left(s, \delta\left(\nu^{s_2-s\pm 1/2}\xi\right)\right) \text{ and } I_\alpha\left(s, \nu^{s_2-s\pm 1/2}\xi \circ \det\right)$$

are reducible. From [47, Theorem 3.1], it is enough to consider only $s > 0$.

7.9 Case (3) computations for $s = 1/2$ and $\chi^2 = 1$, $\chi \neq 1$

We use notations from 7.8.3. If $s = 1/2$ and $\chi^2 = 1$, then by [47, Theorem 3.1] the four terms $I_\gamma(s, \delta(\chi))$ and $I_\gamma(s, \chi \circ \det)$, $\gamma = \alpha$, β all reduces.

If $s = 1/2$ and $\chi^2 = 1$, then

$$I(\nu\chi \otimes \chi) = I(\chi \otimes \nu) = I(\chi\nu \otimes \nu^{-1}) = I(\chi \otimes \nu^{-1}\chi) = I(\nu^{-1}\chi \otimes \nu) = I(\chi \otimes \nu^{-1})$$

in $[\mathrm{R}(G_2)]$ (we omit the terms differ by s_α). Therefore

$$\begin{aligned} I_\beta(\nu^{1/2}\chi \circ \det) + I_\beta\left(\nu^{1/2}\delta(\chi)\right) &= I(\chi \otimes \nu) = I(\nu \otimes \chi) = I(\nu\chi \otimes \chi) \\ &= I_\alpha(\nu^{1/2}\chi \circ \det) + I_\alpha(\nu^{1/2}\delta(\chi)) \ [\mathrm{R}(G_2)]. \end{aligned} \tag{7.9.1}$$

Moreover, since $I^\beta(\chi\nu \otimes \chi) = I^\beta(\nu \otimes \chi)$ $[\mathrm{R}(L_\beta)]$ and they are irreducible, they are isomorphic as representations, and

$$I(\chi\nu \otimes \chi) = I_\beta(I^\beta(\chi\nu \otimes \chi)) = I_\beta(I^\beta(\nu \otimes \chi)) = I(\nu \otimes \chi)$$

as representations. Similarly, we have

$$I(\nu\chi \otimes \chi) \cong I(\chi \otimes \nu) \cong I(\nu \otimes \chi)$$

as representations and they are all isomorphic representations of $I(\nu\chi \otimes \chi)$.

Corollary 7.9.1. *If we take $s = 1/2$, $\chi^2 = 1$ in Lemma 7.7.1, we have in $\mathrm{R}(L_\alpha)$:*

$$r_\alpha\left(I_\alpha\left(1/2, \delta(\chi)\right)\right) = \nu^{1/2}\delta(\chi) + \nu^{-1/2}\delta(\chi) + I^\alpha(\nu \otimes \chi) + I^\alpha(\nu\chi \otimes \nu^{-1}), \tag{7.9.2}$$

$$r_\alpha\left(I_\alpha(1/2, \chi \circ \det)\right) = \nu^{1/2}\chi \circ \det + \nu^{-1/2}\chi \circ \det + I^\alpha(\chi \otimes \nu^{-1}) + I^\alpha(\nu \otimes \nu^{-1}\chi), \tag{7.9.3}$$

$$\begin{aligned} r_\alpha\left(I_\beta\left(1/2, \delta(\chi)\right)\right) &= I^\alpha(\chi \otimes \nu) + I^\alpha(\nu \otimes \nu^{-1}\chi) + I^\alpha(\nu\chi \otimes \chi) \\ &= \nu^{1/2}\delta(\chi) + \nu^{1/2}\chi \circ \det + I^\alpha(\chi \otimes \nu) + I^\alpha(\nu \otimes \nu^{-1}\chi), \end{aligned} \tag{7.9.4}$$

$$\begin{aligned} r_\alpha\left(I_\beta(1/2, \chi \circ \det)\right) &= I^\alpha(\nu\chi \otimes \nu^{-1}) + I^\alpha(\nu^{-1} \otimes \chi) + I^\alpha(\chi \otimes \nu^{-1}\chi) \\ &= \nu^{-1/2}\delta(\chi) + \nu^{-1/2}\chi \circ \det + I^\alpha(\nu\chi \otimes \nu^{-1}) + I^\alpha(\nu^{-1} \otimes \chi). \end{aligned} \tag{7.9.5}$$

Corollary 7.9.2. *If we take $s = 1/2$, $\chi^2 = 1$ in lemma 7.7.2, we have in $\mathrm{R}(L_\beta)$:*

$$r_\beta\left(I_\beta\left(1/2, \delta(\chi)\right)\right) = \nu^{1/2}\delta(\chi) + \nu^{-1/2}\delta(\chi) + I^\beta(\nu \otimes \chi) + I^\beta(\chi \otimes \nu\chi), \tag{7.9.6}$$

$$r_\beta(I_\beta(1/2, \chi\circ\det)) = \nu^{1/2}\chi\circ\det + \nu^{-1/2}\chi\circ\det + I^\beta(\nu^{-1}\otimes\nu\chi) + I^\beta(\nu^{-1}\chi\otimes\chi), \quad (7.9.7)$$

$$\begin{aligned} r_\beta(I_\alpha(1/2,\delta(\chi))) &= I^\beta(\nu\chi\otimes\chi) + I^\beta(\nu^{-1}\otimes\nu\chi) + I^\beta(\chi\otimes\nu) \\ &= \nu^{1/2}\delta(\chi) + \nu^{1/2}\chi\circ\det + I^\beta(\nu\otimes\chi) + I^\beta(\nu^{-1}\otimes\nu\chi), \end{aligned} \quad (7.9.8)$$

$$\begin{aligned} r_\beta\left(I_\alpha\left(1/2,\chi\circ\det\right)\right) &= I^\beta(\chi\otimes\nu\chi) + I^\beta(\nu^{-1}\otimes\chi) + I^\beta(\nu^{-1}\chi\otimes\nu) \\ &= \nu^{-1/2}\delta(\chi) + \nu^{-1/2}\chi\circ\det + I^\beta(\chi\otimes\nu\chi) + I^\beta(\nu^{-1}\otimes\chi), \end{aligned} \quad (7.9.9)$$

In (7.9.9) and (7.9.8), we used [15, Lemma 5.4 (iii)]: $I^\beta(\chi_1\otimes\chi_2) = I^\beta\left(s_\beta(\chi_1\otimes\chi_2)\right)$ *in* $\mathrm{R}(L_\beta)$.

Proposition 7.9.3 (G. Muic Proposition 4.1)**.** *Suppose that* χ *is a character of order* 2. *Then we have the following:*

(1) The induced representation $I(\nu\chi\otimes\chi)$ *has a unique irreducible subrepresentation* $\pi(\chi)$. *We have* $r_\phi(\pi(\chi)) = \nu\chi\otimes\chi + \nu\otimes\chi + \chi\otimes\nu$. $\pi(\chi)$ *is square integrable.*

(2) In $\mathrm{R}(G_2)$, *we have:*

$$\begin{aligned} I_\alpha(1/2,\delta(\chi)) &= \pi(\chi) + J_\alpha(1/2,\delta(\chi)), \\ I_\beta(1/2,\delta(\chi)) &= \pi(\chi) + J_\beta(1/2,\delta(\chi)), \\ I_\alpha(1/2,\chi\circ\det) &= J_\beta(1,\pi(1,\chi)) + J_\beta(1/2,\delta(\chi)), \\ I_\beta(1/2,\chi\circ\det) &= J_\beta(1,\pi(1,\chi)) + J_\alpha(1/2,\delta(\chi)). \end{aligned} \quad (7.9.10)$$

Proof. By a theorem of F. Rodier [54], we know $I(\nu\chi\otimes\chi)$ has length 4, multiplicity 1, thus it contains a unique irreducible subrepresentation, denoted by $\pi(\chi)$.

We know that

$$\begin{aligned} I_\beta(\nu^{1/2}\chi\circ\det) + I_\beta(\nu^{1/2}\delta(\chi)) &= I(\chi\otimes\nu) = I(\nu\otimes\chi) = I(\nu\chi\otimes\chi) \\ &= I_\alpha(\nu^{1/2}\chi\circ\det) + I_\alpha(\nu^{1/2}\delta(\chi)). \end{aligned} \quad (7.9.11)$$

Notice that $\pi(\chi)$ is a (unique) subrepresentation of $I_\beta(\nu^{1/2}\delta(\chi))$ and $I_\alpha(\nu^{1/2}\delta(\chi))$ since they are subrepresentations of $I(\chi\otimes\nu)$ and $I(\nu\chi\otimes\chi)$ respectively. $r_\alpha(\pi(\chi))$ must be contained in the common parts of (7.9.2) and (7.9.4). Since

$$0 \neq \mathrm{Hom}_{G_2}(\pi(\chi), I(\nu\otimes\chi)) = \mathrm{Hom}_{L_\alpha}(r_\alpha(\pi(\chi)), I^\alpha(\nu\otimes\chi)),$$

we know $r_\alpha(\pi(\chi))$ must contain $I^\alpha(\nu\otimes\chi)$. Repeat the above argument with $I(\nu\otimes\chi)$

replaced by $I_\alpha(\nu^{1/2}\delta(\chi))$, $r_\alpha(\pi(\chi))$ must contain $\nu^{1/2}\delta(\chi)$. To satisfy these three conditions, we conclude that

$$r_a(\pi(\chi)) = \nu^{1/2}\delta(\chi) + I^\alpha(\nu \otimes \chi).$$

Composing with $r_T^{L_\alpha}$, we prove (1).

By Langlands quotient theorem applied to the following:

$$I(\chi\nu \otimes \chi) = I(\nu \otimes \chi) = I_\beta(1, I^\beta(1 \otimes \chi)),$$

we see $J_\beta(1, \pi(1, \chi))$ is a composition factor of $I(\chi\nu \otimes \chi)$. The same theorem applied to $I_\alpha(1/2, \delta(\chi))$ and $I_\beta(1/2, \delta(\chi))$, we find the rest two (it has length 4) composition factors: $J_\alpha(1/2, \delta(\chi))$ and $J_\beta(1/2, \delta(\chi))$.

We have

$$\pi(\chi) + J_\alpha(1/2, \delta(\chi)) \subseteq I_\alpha(1/2, \delta(\chi)), \ [\mathrm{R}(G_2)]$$

and we can exclude the possibility that $I_\alpha(1/2, \delta(\chi))$ contains more terms because (7.9.3) is not contained in (7.9.5) or (7.9.4). Using the same argument for β case, we can prove (2).

□

Using similar methods of the proposition above, we can find the image of 4 components under r_α and r_β. We list them in the following propositions:

Proposition 7.9.4. *We have the following:*

$$\begin{aligned}
r_\alpha(\pi(\chi)) &= \nu^{1/2}\delta(\chi) + I^\alpha(\nu \otimes \chi), \ r_\beta(\pi(\chi)) = \nu^{1/2}\delta(\chi) + I^\beta(\nu \otimes \chi), \\
r_\alpha(J_\alpha(1/2, \delta(\chi))) &= \nu^{-1/2}\delta(\chi) + I^\alpha(\nu\chi \otimes \nu^{-1}), \\
r_\beta(J_\alpha(1/2, \delta(\chi))) &= \nu^{1/2}\chi \circ \det + I^\beta(\nu^{-1} \otimes \nu\chi), \\
r_\alpha(J_\beta(1/2, \delta(\chi))) &= \nu^{1/2}\chi \circ \det + I^\alpha(\nu \otimes \nu^{-1}\chi), \\
r_\beta(J_\beta(1/2, \delta(\chi))) &= \nu^{-1/2}\delta(\chi) + I^\beta(\chi \otimes \nu\chi), \\
r_\alpha(J_\beta(1, \pi(1, \chi))) &= \nu^{-1/2}\chi \circ \det + I^\alpha(\chi \otimes \nu^{-1}), \\
r_\beta(J_\beta(1, \pi(1, \chi))) &= \nu^{-1/2}\chi \circ \det + I^\beta(\nu^{-1}\chi \otimes \chi).
\end{aligned} \tag{7.9.12}$$

Proposition 7.9.5. *We compute the Aubert-Zelevinsky duality of all the irreducible representations listed above:*

$$D_{G_2}(\pi(\chi)) = J_\beta(1, \pi(1, \chi)), \quad D_{G_2}(J_\alpha(1/2, \delta(\chi))) = J_\beta(1/2, \delta(\chi)) \ [\mathrm{R}(G_2)] \tag{7.9.13}$$

Proof. We now compute

$$\begin{aligned}
&D_{G_2}(\pi(\chi)) = I \circ r_T^{G_2}(\pi(\chi)) - I_\alpha \circ r_\alpha(\pi(\chi)) - I_\beta \circ r_\beta(\pi(\chi)) + \pi(\chi)\\
&= I(\nu\chi \otimes \chi) + I(\nu \otimes \chi) + I(\chi \otimes \nu) - I_\alpha(\nu^{1/2}\delta(\chi)) - I_\alpha(I^\alpha(\nu \otimes \chi))\\
&- I_\beta(\nu^{1/2}\delta(\chi)) - I_\beta(I^\beta(\nu \otimes \chi)) + \pi(\chi)\\
&= 3I(\nu\chi \otimes \chi) - I_\alpha(\nu^{1/2}\delta(\chi)) - I(\nu \otimes \chi) - I_\beta(\nu^{1/2}\delta(\chi)) - I(\nu \otimes \chi) + \pi(\chi)\\
&= I(\nu\chi \otimes \chi) - I_\alpha(\nu^{1/2}\delta(\chi)) - I_\beta(\nu^{1/2}\delta(\chi)) + \pi(\chi)\\
&= I_\alpha(1/2, \chi \circ \det) - J_\beta(1/2, \delta(\chi))\\
&= J_\beta(1, \pi(1, \chi)).
\end{aligned} \tag{7.9.14}$$

We can either deduce the rest of this proposition directly from [47, Lemma 3.1] or use a similar calculation:

$$\begin{aligned}
&D_{G_2}(J_\alpha(1/2, \delta(\chi))) = I \circ r_\phi(J_\alpha(1/2, \delta(\chi))) - I_\alpha \circ r_\alpha(J_\alpha(1/2, \delta(\chi)))\\
&- I_\beta \circ r_\beta(J_\alpha(1/2, \delta(\chi))) + J_\alpha(1/2, \delta(\chi))\\
&= I \circ r_T^{L_\alpha} \circ \left(\nu^{-1/2}\delta(\chi) + I^\alpha(\nu\chi \otimes \nu^{-1})\right) - I_\alpha \circ \left(\nu^{-1/2}\delta(\chi) + I^\alpha(\nu\chi \otimes \nu^{-1})\right)\\
&- I_\beta \circ \left(\nu^{1/2}\chi \circ \det + I^\beta(\nu^{-1} \otimes \nu\chi)\right) + J_\alpha(1/2, \delta(\chi))\\
&= I(\chi \otimes \nu^{-1}\chi) + I(\nu\chi \otimes \nu^{-1}) + I(\nu^{-1} \otimes \nu\chi) - I_\alpha(\nu^{-1/2}\delta(\chi)) - I(\nu\chi \otimes \nu^{-1})\\
&- I_\beta(\nu^{1/2}\chi \circ \det) - I(\nu^{-1} \otimes \nu\chi) + J_\alpha(1/2, \delta(\chi))\\
&= I(\chi \otimes \nu^{-1}\chi) + I(\nu\chi \otimes \nu^{-1}) + I(\nu^{-1} \otimes \nu\chi) - I_\alpha(\nu^{-1/2}\delta(\chi)) - I(\nu\chi \otimes \nu^{-1})\\
&- I_\beta(\nu^{1/2}\chi \circ \det) - I(\nu^{-1} \otimes \nu\chi) + J_\alpha(1/2, \delta(\chi))\\
&= I(\chi \otimes \nu^{-1}\chi) - I_\alpha(\nu^{-1/2}\delta(\chi)) - I_\beta(\nu^{1/2}\chi \circ \det) + J_\alpha(1/2, \delta(\chi))\\
&= I_\alpha(1/2, \chi \circ \det) - I_\beta(1/2, \chi \circ \det) + J_\alpha(1/2, \delta(\chi))\\
&= J_\beta(1, \pi(1, \chi)) + J_\beta(1/2, \delta(\chi)) - (J_\beta(1, \pi(1, \chi)) + J_\alpha(1/2, \delta(\chi))) + J_\alpha(1/2, \delta(\chi))\\
&= J_\beta(1/2, \delta(\chi)).
\end{aligned} \tag{7.9.15}$$

In the steps of above computations, we need to use [15, Lemma 5.4 (iii)] to get

$$I_\alpha(\nu^{-1/2}\delta(\chi)) = I_\alpha(\nu^{1/2}\delta(\chi))$$

in $\mathrm{R}(G_2)$, and [14, Geometric Lemma] for computing $r_T^{L_\alpha} I^\alpha(\nu\chi \otimes \nu^{-1})$. □

7.9.1 Case (3) within the case $\mathfrak{s} = [T, \xi \otimes \xi]_G$: χ ramified quadratic

If ξ_2 is ramified quadratic, then by the discussion in [9, Section 9.3.2], we have $\mathfrak{J}^\mathfrak{s} = \mathrm{SO}_4(\mathbb{C})$ and $\mathcal{H}^\mathfrak{s} = \mathcal{H}(\mathrm{SO}_4(F), 1)$, we know that $\mathrm{SO}_4(\mathbb{C}) \cong \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})/\{\pm 1\}$. Using [50, section 3 table 2.1], we can index the modules by two triples of type A_1:

We get the proposition for modules of Hecke algebra:

Proposition 7.9.6. *Using the indexing triples to denote the standard modules, we have:*

$$\begin{aligned} D_{\mathcal{H}^{\mathfrak{s}}}(M_{(t_a,e_{\alpha_1},1),(t_a,e_{\alpha_1},1)}) &= M_{(t_a,0,1),(t_a,0,1)}, \\ D_{\mathcal{H}^{\mathfrak{s}}}(M_{(t_a,0,1),(t_a,e_{\alpha_1},1)}) &= M_{(t_a,e_{\alpha_1},1),(t_a,0,1)}. \end{aligned} \tag{7.9.16}$$

7.9.2 Case (3) within the case $\mathfrak{s} = [T, 1]_G$: χ unramified quadratic

If ξ_2 is unramified quadratic, $\mathfrak{J}^{\mathfrak{s}} = G_2(\mathbb{C})$, and $\mathcal{H}^{\mathfrak{s}} = \mathcal{H}(G_2(F), 1)$. Then by [9, table 16], we get the proposition for modules of Hecke algebra:

Proposition 7.9.7. *Using the indexing triples to denote the standard modules, we have:*

$$\begin{aligned} D_{\mathcal{H}^{\mathfrak{s}}}(M_{t_d,e_{\alpha^\vee}+e_{\alpha^\vee+2\beta^\vee},1}) &= M_{t_d,0,1}, \\ D_{\mathcal{H}^{\mathfrak{s}}}(M_{t_d,e_{\alpha^\vee},1}) &= M_{t_d,e_{2\beta^\vee+\alpha^\vee},1}. \end{aligned} \tag{7.9.17}$$

7.10 Case (3) computations for $s = 1/2$ and $\chi^3 = 1$, $\chi \neq 1$

If $s = 1/2$ and $\chi^3 = 1$, then by [47, Theorem 3.1], $I_\alpha(s, \delta(\chi))$ and $I_\alpha(s, \chi \circ \det)$ reduce, while $I_\beta(s, \delta(\chi))$ or $I_\beta(s, \chi \circ \det)$ do not.

If $s = 1/2$ and $\chi^3 = 1$, then

$$\begin{aligned} I(\nu\chi \otimes \chi) = I(\nu\chi^{-1} \otimes \nu^{-1}\chi^{-1}) &= I(\chi \otimes \chi\nu^{-1}) = I(\nu^{-1}\chi^{-1} \otimes \chi^{-1}) \\ &= I(\nu\chi^{-1} \otimes \chi^{-1}) = I(\nu\chi \otimes \nu^{-1}\chi) \ [\mathrm{R}(G_2)] \end{aligned}$$

(we omit the terms differ by s_α). We deduce

$$\begin{aligned} I_\alpha(\nu^{-1/2}\chi^{-1} \circ \det) + I_\alpha(\nu^{-1/2}\delta(\chi^{-1})) &= I_\alpha(\nu^{1/2}\chi^{-1} \circ \det) + I_\alpha(\nu^{1/2}\delta(\chi^{-1})) \\ &= I_\alpha(\nu^{1/2}\chi \circ \det) + I_\alpha(\nu^{1/2}\delta(\chi)) \ [\mathrm{R}(G_2)] \end{aligned}$$

from

$$I(\nu^{-1}\chi^{-1} \otimes \chi^{-1}) = I(\nu\chi^{-1} \otimes \chi^{-1}) = I(\nu\chi \otimes \chi) \ [\mathrm{R}(G_2)]$$

respectively.

Moreover, since

$$I^\beta(\nu\chi \otimes \chi) = I^\beta\left(s_\beta \circ (\nu\chi \otimes \chi)\right) = I^\beta(\nu\chi^{-1} \otimes \chi^{-1}) \ [\mathrm{R}(L_\beta)]$$

and they are irreducible, they are thus isomorphic as representations. We have

$$I(\nu\chi \otimes \chi) = I_\beta(I^\beta(\nu\chi \otimes \chi)) \cong I_\beta\left(I^\beta(\nu\chi^{-1} \otimes \chi^{-1})\right) = I(\nu\chi^{-1} \otimes \chi^{-1})$$

as representations.

Corollary 7.10.1. *If we take $\chi^3 = 1$, $s = 1/2$ in Lemma 7.7.1, we have:*

$$\begin{aligned} r_\alpha\left(I_\alpha\left(1/2, \delta\left(\chi\right)\right)\right) &= \nu^{1/2}\delta(\chi) + \nu^{-1/2}\delta(\chi^{-1}) + I^\alpha(\nu\chi^{-1} \otimes \chi^{-1}) + I^\alpha(\nu\chi \otimes \nu^{-1}\chi) \\ &= \nu^{1/2}\delta(\chi) + \nu^{-1/2}\delta(\chi^{-1}) + \nu^{1/2}\chi^{-1} \circ \det + \nu^{1/2}\delta(\chi^{-1}) \\ &+ I^\alpha(\nu\chi \otimes \nu^{-1}\chi), \end{aligned} \tag{7.10.1}$$

$$\begin{aligned} r_\alpha\left(I_\alpha(1/2, \chi \circ \det)\right) &= \nu^{1/2}\chi \circ \det + \nu^{-1/2}\chi^{-1} \circ \det + I^\alpha(\chi \otimes \nu^{-1}\chi) + I^\alpha(\nu\chi^{-1} \otimes \nu^{-1}\chi^{-1}) \\ &= \nu^{1/2}\chi \circ \det + \nu^{-1/2}\chi^{-1} \circ \det + \nu^{-1/2}\delta(\chi) + \nu^{-1/2}\chi \circ \det \\ &+ I^\alpha(\nu\chi^{-1} \otimes \nu^{-1}\chi^{-1}). \end{aligned} \tag{7.10.2}$$

If we replace χ by χ^{-1}, then we have:

$$\begin{aligned} r_\alpha\left(I_\alpha\left(1/2, \delta(\chi^{-1})\right)\right) &= \nu^{1/2}\delta(\chi^{-1}) + \nu^{-1/2}\delta(\chi) + \nu^{1/2}\chi \circ \det + \nu^{1/2}\delta(\chi) \\ &+ I^\alpha(\nu\chi^{-1} \otimes \nu^{-1}\chi^{-1}), \end{aligned} \tag{7.10.3}$$

$$\begin{aligned} r_\alpha\left(I_\alpha\left(1/2, \chi^{-1} \circ \det\right)\right) &= \nu^{1/2}\chi^{-1} \circ \det + \nu^{-1/2}\chi \circ \det + \nu^{-1/2}\delta(\chi^{-1}) + \nu^{-1/2}\chi^{-1} \circ \det \\ &+ I^\alpha(\nu\chi \otimes \nu^{-1}\chi). \end{aligned} \tag{7.10.4}$$

Corollary 7.10.2. *If we take $\chi^3 = 1$, $s = 1/2$ in (7.8.27) and (7.8.28) of Lemma 7.7.2, we have:*

$$r_\beta(I_\alpha(1/2, \delta(\chi))) = I^\beta(\nu\chi \otimes \chi) + I^\beta(\nu^{-1}\chi \otimes \nu\chi) + I^\beta(\chi^{-1} \otimes \nu\chi^{-1}), \tag{7.10.5}$$

$$r_\beta\left(I_\alpha(1/2, \chi \circ \det)\right) = I^\beta(\chi \otimes \nu\chi) + I^\beta(\nu^{-1}\chi \otimes \chi) + I^\beta(\nu^{-1}\chi^{-1} \otimes \nu\chi^{-1}). \tag{7.10.6}$$

If we replace χ by χ^{-1}, then:

$$r_\beta\left(I_\alpha\left(1/2, \delta(\chi^{-1})\right)\right) = I^\beta(\nu\chi \otimes \chi) + I^\beta(\nu^{-1}\chi^{-1} \otimes \nu\chi^{-1}) + I^\beta(\chi \otimes \nu\chi), \tag{7.10.7}$$

$$r_\beta\left(I_\alpha\left(1/2, \chi^{-1} \circ \det\right)\right) = I^\beta(\chi^{-1} \otimes \nu\chi^{-1}) + I^\beta(\nu^{-1}\chi \otimes \chi) + I^\beta(\nu^{-1}\chi \otimes \nu\chi). \tag{7.10.8}$$

In equation (7.10.7) and (7.10.8) we used $I^\beta\left(s_\beta(\chi_1 \otimes \chi_2)\right) = I^\beta(\chi_1 \otimes \chi_2)$ to the first term and second term respectively.

Proposition 7.10.3 (G. Muic Proposition 4.2)**.** *If a character χ satisfies $\chi^3 = 1$, then we have the following*

1. *The induced representation $I(\nu\chi \otimes \chi)$ has a unique subrepresentation $\pi(\chi)$. We have*

$$r_\varnothing(\pi(\chi)) = \nu\chi \otimes \chi + \nu\chi^{-1} \otimes \chi^{-1},$$

and $\pi(\chi) \cong \pi(\chi^{-1})$.

2. In $\mathrm{R}(G_2)$*, we have:*

$$\begin{aligned} I_\alpha\left(1/2, \delta(\chi)\right) &= \pi(\chi) + J_\alpha\left(1/2, \delta(\chi)\right), \\ I_\alpha\left(1/2, \chi \circ \det\right) &= J_\beta\left(1, \pi(\chi^{-1}, \chi^{-1})\right) + J_\alpha\left(1/2, \delta(\chi^{-1})\right), \\ I_\alpha\left(1/2, \delta(\chi^{-1})\right) &= \pi(\chi) + J_\alpha(1/2, \delta(\chi^{-1})), \\ I_\alpha\left(1/2, \chi^{-1} \circ \det\right) &= J_\beta\left(1, \pi(\chi^{-1}, \chi^{-1})\right) + J_\alpha\left(1/2, \delta(\chi)\right). \end{aligned} \tag{7.10.9}$$

Remark. We point out that there is a typo in [47, Proposition 4.2], the $J_\beta\left(1, \pi(\chi, \chi^{-1})\right)$ should be replaced by $J_\beta\left(1, \pi(\chi^{-1}, \chi^{-1})\right)$ instead.

Proof. By a theorem of F. Rodier [54], we know $I(\nu\chi \otimes \chi)$ has length 4, multiplicity 1, thus it contains a unique irreducible subrepresentation, denoted by $\pi(\chi)$.

Notice that $\pi(\chi)$ is a (unique) subrepresentation of $I_\alpha(\nu^{1/2}\delta(\chi^{-1}))$ and $I_\alpha\left(\nu^{1/2}\delta(\chi)\right)$ since they are subrepresentations of $I(\nu\chi^{-1} \otimes \chi^{-1})$ and $I(\nu\chi \otimes \chi)$ respectively. $r_\alpha(\pi(\chi))$ must be contained in the common parts of (7.10.1) and (7.10.3). Since

$$0 \neq \mathrm{Hom}_{G_2}(\pi(\chi), I_\alpha(\nu^{1/2}\delta(\chi^{-1}))) = \mathrm{Hom}_{L_\alpha}(r_\alpha(\pi(\chi)), \nu^{1/2}\delta(\chi^{-1})),$$

we know $r_\alpha(\pi(\chi))$ must contain $\nu^{1/2}\delta(\chi^{-1})$. Repeat the above argument with $I(\nu\chi \otimes \chi)$ replaced by $I(\nu\chi^{-1} \otimes \chi^{-1})$, $r_\alpha(\pi(\chi))$ must contain $\nu^{1/2}\delta(\chi)$. To satisfy these three conditions, we conclude that

$$r_\alpha(\pi(\chi)) = \nu^{1/2}\delta(\chi) + \nu^{1/2}\delta(\chi^{-1}).$$

Composing with $r_T^{L_\alpha}$, we prove (1).

Applying Langlands quotient theorem to the following:

$$I(\nu\chi \otimes \chi) = I(\nu\chi^{-1} \otimes \chi^{-1}) = I_\beta(1, \pi(\chi^{-1}, \chi^{-1})),$$

we see $J_\beta(1, \pi(\chi^{-1}, \chi^{-1}))$ is a composition factor of $I(\nu\chi \otimes \chi)$. The same theorem applied to $I_\alpha(1/2, \delta(\chi))$ and $I_\alpha(1/2, \delta(\chi^{-1}))$, we find the rest two (it has length 4) composition factors: $J_\alpha(1/2, \delta(\chi))$ and $J_\alpha(1/2, \delta(\chi^{-1}))$.

We have $\pi(\chi) + J_\alpha(1/2, \delta(\chi)) \subseteq I_\alpha(1/2, \delta(\chi))$, and we can exclude the possibility that $I_\alpha(1/2, \delta(\chi))$ contains more terms because (7.10.2) is not contained in (7.10.4) or (7.10.3). Using the same argument for χ^{-1} case, we can prove (2).

$\square$

Using similar methods of the proposition above, we can find the image of 4 components under r_α and r_β. We list them in the following proposition:

Proposition 7.10.4. *Based on the proof of Proposition 7.10.3, we have the following*

$$\begin{aligned}
r_\alpha(\pi(\chi)) &= \nu^{1/2}\delta(\chi) + \nu^{1/2}\delta(\chi^{-1}), \quad r_\beta(\pi(\chi)) = I^\beta(\nu\chi\otimes\chi),\\
r_\alpha(J_\alpha(1/2,\delta(\chi))) &= \nu^{-1/2}\delta(\chi^{-1}) + \nu^{1/2}\chi^{-1}\circ\det + I^\alpha(\nu\chi\otimes\nu^{-1}\chi),\\
r_\beta(J_\alpha(1/2,\delta(\chi))) &= I^\beta(\nu^{-1}\chi\otimes\nu\chi) + I^\beta(\chi^{-1}\otimes\nu\chi^{-1}),\\
r_\alpha(J_\alpha(1/2,\delta(\chi^{-1}))) &= \nu^{-1/2}\delta(\chi) + \nu^{1/2}\chi\circ\det + I^\alpha(\nu\chi^{-1}\otimes\nu^{-1}\chi^{-1}),\\
r_\beta(J_\alpha(1/2,\delta(\chi^{-1}))) &= I^\beta(\nu^{-1}\chi^{-1}\otimes\nu\chi^{-1}) + I^\beta(\chi\otimes\nu\chi),\\
r_\alpha(J_\beta(1,\pi(\chi^{-1},\chi^{-1}))) &= \nu^{-1/2}\chi^{-1}\circ\det + \nu^{-1/2}\chi\circ\det,\\
r_\beta(J_\beta(1,\pi(\chi^{-1},\chi^{-1}))) &= I^\beta(\nu^{-1}\chi\otimes\chi).
\end{aligned} \tag{7.10.10}$$

Proposition 7.10.5. *We compute the Aubert-Zelevinsky duality of all the irreducible representations listed above:*

$$D_{G_2}(\pi(\chi)) = J_\beta(1,\pi(\chi^{-1},\chi^{-1})), \quad D_{G_2}(J_\alpha(1/2,\delta(\chi))) = J_\alpha(1/2,\delta(\chi^{-1}))\ [\mathrm{R}(G_2)] \tag{7.10.11}$$

Proof. We compute as follows:

$$\begin{aligned}
&D_{G_2}(\pi(\chi)) = I\circ r_T^{G_2}(\pi(\chi)) - I_\alpha\circ r_\alpha(\pi(\chi)) - I_\beta\circ r_\beta(\pi(\chi)) + \pi(\chi)\\
&= I(\nu\chi\otimes\chi) + I(\nu\chi^{-1}\otimes\chi^{-1}) - I_\alpha\left(\nu^{1/2}\delta(\chi)\right) - I_\alpha\left(\nu^{1/2}\delta(\chi^{-1})\right)\\
&- I_\beta\left(I^\beta(\nu\chi\otimes\chi)\right) + \pi(\chi)\\
&= I_\alpha(1/2,\chi\circ\det) - I_\alpha(1/2,\delta(\chi^{-1})) + \pi(\chi)\\
&= J_\beta(1,\pi(\chi^{-1},\chi^{-1})) + J_\alpha(1/2,\delta(\chi^{-1})) - \pi(\chi) - J_\alpha(1/2,\delta(\chi^{-1})) + \pi(\chi)\\
&= J_\beta(1,\pi(\chi^{-1},\chi^{-1})).
\end{aligned} \tag{7.10.12}$$

Similarly, we have

$$
\begin{aligned}
&D_{G_2}\left(J_\alpha\left(1/2,\delta(\chi)\right)\right) = I\circ r_T^{G_2}\left(J_\alpha\left(1/2,\delta(\chi)\right)\right) - I_\alpha\circ r_\alpha\left(J_\alpha\left(1/2,\delta(\chi)\right)\right)\\
&- I_\beta\circ r_\beta\left(J_\alpha\left(1/2,\delta(\chi)\right)\right) + J_\alpha\left(1/2,\delta(\chi)\right)\\
&= I\circ r_T^{L_\alpha}(\nu^{-1/2}\delta(\chi^{-1})) + I\circ r_T^{L_\alpha}(\nu^{1/2}\chi^{-1}\circ\det) + I\circ r_T^{L_\alpha}(I^\alpha(\nu\chi\otimes\nu^{-1}\chi))\\
&- I_\alpha(\nu^{-1/2}\delta(\chi^{-1})) - I_\alpha(\nu^{1/2}\chi^{-1}\circ\det) - I_\alpha(I^\alpha(\nu\chi\otimes\nu^{-1}\chi))) - I_\beta(I^\beta(\nu^{-1}\chi\otimes\nu\chi))\\
&- I_\beta(I^\beta(\chi^{-1}\otimes\nu\chi^{-1})) + J_\alpha\left(1/2,\delta(\chi)\right)\\
&= I\left(\chi^{-1}\otimes\nu^{-1}\chi^{-1}\right) + I\left(\chi^{-1}\otimes\nu\chi^{-1}\right) + 2I\left(\nu\chi\otimes\nu^{-1}\chi\right) - I_\alpha(1/2,\delta(\chi))\\
&- I_\alpha(1/2,\chi^{-1}\circ\det) - I(\nu\chi\otimes\nu^{-1}\chi) - I(\nu^{-1}\chi\otimes\nu\chi) - I(\chi^{-1}\otimes\nu\chi^{-1})\\
&+ J_\alpha\left(1/2,\delta(\chi)\right)\\
&= I\left(\chi^{-1}\otimes\nu^{-1}\chi^{-1}\right) - I_\alpha(1/2,\delta(\chi)) - I_\alpha(1/2,\chi^{-1}\circ\det) + J_\alpha\left(1/2,\delta(\chi)\right)\\
&= J_\beta(1,\pi(\chi^{-1},\chi^{-1})) + J_\alpha(1/2,\delta(\chi^{-1})) - J_\beta\left(1,\pi(\chi^{-1},\chi^{-1})\right) - J_\alpha\left(1/2,\delta(\chi)\right)\\
&+ J_\alpha\left(1/2,\delta(\chi)\right)\\
&= J_\alpha(1/2,\delta(\chi^{-1})).
\end{aligned}
\tag{7.10.13}
$$

□

7.10.1 Case (3) within the case $\mathfrak{s} = [T,\xi\otimes\xi]_G$: χ ramified cubic

If χ is ramified cubic, $\mathfrak{J}^{\mathfrak{s}} = \mathrm{SL}_3(\mathbb{C})$, and $\mathcal{H}^{\mathfrak{s}} = \mathcal{H}(\mathrm{PGL}_3(F),1)$. By [9, table 20] and [50, table 4.1], we get the proposition for modules of Hecke algebra:

Proposition 7.10.6. *Using the indexing triples to denote the standard modules, we have:*

$$
\begin{aligned}
D_{\mathcal{H}^{\mathfrak{s}}}(M_{t_a,e_{\alpha^\vee}+e_{2\alpha^\vee+3\beta^\vee},1}) &= M_{t_a,0,1},\\
D_{\mathcal{H}^{\mathfrak{s}}}(M_{t_a,e_{\alpha^\vee},1}) &= M_{t_a,e_{3\beta^\vee+2\alpha^\vee},1}.
\end{aligned}
\tag{7.10.14}
$$

7.10.2 Case (3) within the case $\mathfrak{s} = [T,1]_G$: χ unramified cubic

If χ is unramified cubic, $\mathfrak{J}^{\mathfrak{s}} = G_2(\mathbb{C})$, and $\mathcal{H}^{\mathfrak{s}} = \mathcal{H}(G_2(F),1)$ by [9, table 19] and [50, table 6.2]. We get the proposition for modules of Hecke algebra:

Proposition 7.10.7. *Using the indexing triples to denote the standard modules, we have:*

$$
\begin{aligned}
D_{\mathcal{H}^{\mathfrak{s}}}(M_{t_c,e_{\alpha^\vee}+e_{\alpha^\vee+3\beta^\vee},1}) &= M_{t_c,0,1},\\
D_{\mathcal{H}^{\mathfrak{s}}}(M_{t_c,e_{\alpha^\vee},1}) &= M_{t_c,e_{3\beta^\vee+\alpha^\vee},1}.
\end{aligned}
\tag{7.10.15}
$$

Appendix A

Homology Representations

This Appendix is devoted to topology background and the construction of homology representations that we need for Section 5.7.

A.1 The topology vocabularies

An (abstract) finite *simplicial complex* Σ consists of a finite set of points $V(\Sigma) := \{x_0, x_1, \dots x_N\}$ (for some $N \in \mathbb{N}$) called *vertices*, together with certain finite, non-empty sets of vertices $\{\sigma\}$ called *simplices*, satisfying the axioms that

(i) each singleton set $\{x\}$ is a simplex,

(ii) each non-empty subset σ' of a simplex σ a is also a simplex.

A simplicial map $f : \Sigma \to \Sigma'$ of simplicial complexes is a map f from the vertices of Σ to those of Σ', such that if $\sigma = \{x_0, x_1, x_2, \dots, x_r\}$ is an r-simplex (*i.e.* simplex consisting of $r+1$ vertices) in Σ, then $\{f(x_0), f(x_1), ..., f(x_r)\}$ are the vertices (possibly with repetitions) of a simplex in Σ'. The finite simplicial complexes form a category, in which morphisms are simplicial maps. We call a subset A of a simplex σ, a *face* of σ. A *subcomplex* of Σ is a collection of subsets in Σ which is a simplicial complex in its own right, with the vertex set being some subset of $V(\Sigma)$.

We give a more useful definition of simplicial complex via posets:

Definition A.1.1 (Simplicial complex)**.** Any poset Σ satisfying the properties below is referred to as a *simplicial complex*:

(a) Any two elements A, $B \in \Sigma$, have a greatest lower bound, denoted by $A \cap B$.

(b) For $A \in \Sigma$, the poset of all faces $\Sigma_{\leq A}$ is isomorphic to $2^{\{1,2,\dots,r\}}$ for some positive integer r with inclusion ordering.

Now let G be a finite group, a simplicial complex Σ is called a *G-complex* and G is said to act on Σ if the vertices of Σ form a G-set and if the action of G carries simplices to simplices. The finite G-complexes form a category, in which the objects are G-complexes, and the morphisms are simplicial maps preserving the G-action.

Definition A.1.2. To each simplicial complex Σ we associate an underlying topological space, denoted by $|\Sigma|$, consisting of all real valued functions p such that:

(i) $p(x) \geqslant 0$ for all vertices in Σ,

(ii) $\sum_{x \in V(\Sigma)} p(x) = 1$,

(iii) $\operatorname{supp} p(x) := \{x \in V(\Sigma) \mid p(x) \neq 0\}$ is a simplex of Σ.

We now describe the topology structure of $|\Sigma|$. For each n-simplex $\sigma = \{x_0, x_1, \ldots, x_n\}$ of Σ, denote by $|\sigma|$ the subset of $|\Sigma|$ defined by

$$|\sigma| = \{p \in |\Sigma| \mid \operatorname{supp} p \subseteq \sigma\}.$$

Each point $p \in |\sigma|$ determines a point $\sum_{i=1}^{n} p(x_i) e_i$ in $\mathbb{R}^n$, where $\{e_i\}_{i=1}^{n}$ is an orthonormal basis of $\mathbb{R}^n$. In view of the condition (ii) above, we give a bijection between $|\sigma|$ and the standard n-simplex $|\{0, e_1, e_2, \ldots, e_n\}|$ of $\mathbb{R}^n$, and topologize $|\sigma|$ via this bijection. Finally we topologize Σ by the family of subsets U such that $U \cap |\sigma|$ is open in $|\sigma|$ for every simplex σ of Σ. A simplicial map $f : \Sigma \to \Sigma'$ yields a continuous map $|f| : |\Sigma| \to |\Sigma'|$, defined by

$$p = \sum_{x \in V(\Sigma)} p(x) x \mapsto |f| p = \sum_{x \in V(\Sigma)} p(x) f(x) \in |\Sigma'|.$$

We can talk about things like closure, dimension of a simplex using the above topological description, for example, a r-simplex is of dimension r, we call a simplex of maximal dimension a *chamber*.

A finite group G is said to act on a chain complex $C_\bullet = (C_r, \partial_r)_{r \geqslant 0}$ ($\partial_r C_r \subseteq C_{r-1}$, $\partial_0 C_0 = 0$ and $\partial^2 = 0$) if each subspace C_r is a finite dimensional KG-module and if ∂ commutes with the action of G. We refer to $C_\bullet$ as a chain complex with G-action in this case, and then the homology spaces $\{H_r(C_\bullet)\}$ are KG-modules, affording what we shall call *homology representations* of G.

Now let Σ be a G-complex, and K a field. We now define a chain complex $C_\bullet(\Sigma)$ with G-action. For $r \geqslant 0$, the subspace $C_r(\Sigma)$ of r-chains has a K-basis $\{c_\sigma\}$, indexed by a r-simplices σ in Σ. The action of G is given by its permutation action on the r-chains:

$$g c_\sigma = c_{g\sigma}, \qquad \text{for } \sigma \text{ as above, and } g \in G.$$

The boundary homomorphism $\partial_r : C_r(\Sigma) \to C_{r-1}(\Sigma)$ is given by

$$\partial c_\sigma = \sum_{i=0}^{r} (-1)^i c_{\sigma_i},$$

where if $\sigma = \{x_0, \cdots, x_r\}$, then $\sigma_i = \{x_0, \cdots, \hat{x}_i, \cdots, x_r\}$ is the i-th face of σ, for $0 \leqslant i \leqslant r$ with x_i omitted. The vector space $C_r(\Sigma)$ is called the *vector space of r-chains*, where an r-chain is a formal linear combination of r-simplices with coefficients in K. We can verify that $\partial^2 = 0$, hence $C_\bullet(\Sigma)$ is a chain complex with G-action, and the resulting homology representation

$$H_*(C_\bullet(\Sigma)) = \bigoplus_{r=0}^{\infty} H_r(C_\bullet(\Sigma))$$

is called *the homology representation* associated with the G-complex Σ.

Definition A.1.3. Let Σ be a G-complex. The Lefschetz character of the homology representation $H_*(C_\bullet(\Sigma))$ is the map $\mathcal{X} : G \to K$ defined by

$$\mathcal{X}(g) = \sum_{r \geqslant 0} (-1)^r \operatorname{Tr}\left(g, H_r(C_\bullet(\Sigma))\right).$$

The degree $\mathcal{X}(1)$ of the Lefschetz character is called the *Euler characteristic* of Σ, and is denoted by $\chi(\Sigma)$.

In the case K is a field of characteristic zero the Euler characteristic coincides with the alternating sum

$$\sum_{r=0}^{\infty} (-1)^r \dim H_r(C_\bullet(\Sigma)),$$

which is the *(usual) Euler characteristic* $\chi(|\Sigma|)$ of the underlying topological space $|\Sigma|$.

Proposition A.1.4 (Hopf Trace Formula.). *Let $C_\bullet = \bigoplus_{r>0} C_r$ be a chain complex over a field K, with boundary map ∂, such that each subspace C_r is finite dimensional over K, and $C_r = 0$ for all sufficiently large r. Let $f : C_\bullet \to C_\bullet$ be a* graded chain map of degree zero. *(*i.e. *a K-endomorphism $f : C_\bullet \to C_\bullet$ such that $f(C_r) \subseteq C_r$ for each $r \geqslant 0$, and $f\partial = \partial f$.) Then f induces a K-endomorphism f_* of $H_*(C_\bullet)$ such that $f_*(H_r(C_\bullet)) \subseteq H_r(C_\bullet)$ for all r, and we have*

$$\sum_{r=0}^{\infty} (-1)^r \operatorname{Tr}(f, C_r) = \sum_{r=0}^{\infty} (-1)^r \operatorname{Tr}(f_*, H_r(C)).$$

The G-action on $C_\bullet$ is a graded chain map of degree zero, hence we have

Corollary A.1.5. *Let K be a field of characteristic zero. Then the Euler characteristic $\chi(\Sigma)$ of a G-poset Σ is given by*

$$\chi(\Sigma) = \sum_{r=0}^{\infty} (-1)^r \dim C_r(\Sigma).$$

We now have:

Proposition A.1.6. *Let Σ be a G-complex, and $\mathcal{X}$ the Lefschetz character of the homology representation $H_*(C_\bullet(\Sigma))$ of G. Then we have*

$$\mathcal{X}(g) = \chi(\Sigma^g) = \chi(|\Sigma|^g), \qquad \textit{for each } g \in G.$$

where $|\Sigma|^g$ is the fixed point set under the action of g, and $\chi(|\Sigma|)$ is the usual Euler characteristic introduced above.

A.2 The Coxeter complex

Let R be a root system in an n-dimensional euclidean space $E = (V, (\cdot,\cdot))$ ($\dim V = n$), and let W be the finite Weyl group associated with R. The inner product $(\cdot,\cdot)$ corresponds to the perfect pairing in Section 3.2. Since $W \subseteq O(E)$, W acts on the unit sphere S^{n-1}. Let Δ be a fundamental system in R, it determines the set of positive roots R^+ of R as the elements in R that can be written as $\mathbb{Z}^{\geqslant 0}$-linear combinations of elements of Δ.

For each root $\alpha \in R^+$, we define

$$\begin{aligned} H_\alpha^+ &= \{v \in V \mid (\alpha, v) > 0\}, \\ H_\alpha^- &= \{v \in V \mid (\alpha, v) < 0\}, \\ H_\alpha = H_\alpha^0 &= \{v \in V \mid (\alpha, v) = 0\}. \end{aligned}$$

Definition A.2.1. The Coxeter complex is the collection of all subsets of V of the form

$$\bigcap_{\alpha \in R^+} H_\alpha^{\epsilon_\alpha}, \quad \epsilon_\alpha = +, -, \text{ or } 0.$$

The simplices of the Coxeter complex contained in the closure of the fundamental chamber (*i.e.* $C := \bigcap_{\alpha \in R^+} H_\alpha^+$) are those of the form

$$C_I = \left\{ v \in V \mid \begin{array}{l} (v, \alpha) = 0 \text{ for } \alpha \in I \\ (v, \alpha) > 0 \text{ for } \alpha \in \Delta - I \end{array} \right\},$$

where I is any subset of Δ.

We have an operation of the Weyl group W on the Coxeter complex. From [21, Proposition 2.6.1., Proposition 2.6.2. and Proposition 2.6.3.], we sumarize:

Proposition A.2.2. *(i) For any subset I of Δ, the stabilizer of C_I in W is the standard parabolic subgroup W_I associated with I. It is also the point-wise stabilizer of all vectors in C_I.*

(ii) Each simplex of the Coxeter complex can be transformed into exactly one C_I by an element of the Weyl group.

(iii) *The parabolic subgroups of* W *are the stabilizers in* W *of the simplices of the Coxeter complex,* i.e. *the stabilizers of* wC_I *are the parabolic subgroups* ${}^{w}W_I$.

Thus we have the following Corollary

Corollary A.2.3. *The map* $wC_I \to wW_I, w \in W, I \subset \Delta$, *is a bijection from the set of faces of the closed chambers* $\{w\overline{C}\}_{w\in W}$ *to the set of left cosets of the parabolic subgroup* W_I. *Moreover,*

$$wC_I \subseteq w'C_{I'} \Leftrightarrow wW_I \supseteq w'W_{I'}, \qquad \textit{for } w, w' \in W \qquad \textit{and } I, I' \subset \Pi.$$

Definition A.2.4 (Coxeter Complex defined via posets)**.** Let (W, S) be a finite Coxeter system, with $\mathrm{Card}(S) \geqslant 2$. The *Coxeter poset* X is the W-poset consisting of all left cosets

$$\{wW_I : w \in W, \quad I \subset S\},$$

ordered by the inverse inclusion. The action of W on X is given by left translation: for $x \in W$, the action is

$$wW_I \mapsto xwW_I \quad \text{for all } w \in W, \quad I \subset S.$$

We see that such defined poset satisfies Definition A.1.1, thus we can see X as a complex. Moreover, it gives us the same complex as Definition A.2.1 using Corollary A.2.3.

Theorem A.2.5. *Let* R *be a root system in* V, *and* (W, S) *the Coxeter system associated with* R, *and* S *is the set of reflections associated with* Δ. *Recall that the dimension of* V *equals* $\mathrm{S} = n \geqslant 2$. *Let* X *be the Coxeter poset associated with* (W, S). *We have the following:*

(i) *The Weyl group acts on the unit sphere* S^{n-1}, *and there is a* W*-equivalent homeomorphism*

$$|X| \cong S^{n-1}$$

where $|X|$ *is defined in Definition A.1.2.*

(ii) *The homology representation of* W *on* $H_*(C_\bullet(X))$ *over the rational field* $\mathbb{Q}$ *is as follows:*

$$H_i(C_\bullet(X)) = \begin{cases} 1_W, & \textit{if } i = 0, \\ \mathrm{sgn}_W, & \textit{if } i = n-1, \\ 0, & \textit{otherwise.} \end{cases}$$

where the sign representation is defined by $\mathrm{sgn}_W(w) = \det w$ *for all* $w \in W$, *and* $1_W(w) = 1$ *for all* $w \in W$.

Proof. (i) Since W is a group of orthogonal transformations, W clearly acts on the sphere S^{n-1}. We now give an outline of the proof that $|X| \cong S^{n-1}$. Let $\overline{C}$ be the chamber in V defined by Δ. It is easily verified, using the fact that Δ is a basis of V, that the hyperplanes H_α, $\alpha \in \Delta$ are the walls of $\overline{C}$, in the sense that $H_\alpha \cap \overline{C}$ is contained in the boundary of $\overline{C}$, and generates the vector space H_α. The same argument shows that each face $\overline{C} \cap H_{\alpha_{i_1}} \cap \cdots \cap H_{\alpha_{i_g}}$, where $I = \{\alpha_{i_1}, \ldots, \alpha_{i_g}\} \subset \Delta$, generates a subspace of V of dimension $n - |I|$. Using the discussion in the previous section, it is then readily shown that the intersection $\overline{C} \cap S^{n-1}$ is homeomorphic to the underlying topological space of an (abstract) $(n-1)$-simplex σ. The vertices of σ correspond to the intersections $C_I \cap S^{n-1}$, for $|I| = n-1, I \subset \Delta$, since for such I, each face C_I is a half-line and intersects S^{n-1} in a point. These points support a spherical simplex homeomorphic to $|\sigma|$.

By Proposition A.2.2, S^{n-1} is the union of the intersections $\{w\overline{C} \cap S^{n-1}\}_{w \in W}$, and their interiors partition the set $S^{n-1} - \bigcup_{\alpha \in \Delta} H_\alpha$. By the first paragraph, each such intersection is homeomorphic to $|\sigma|$. These intersections $\{w\overline{C} \cap S^{n-1}\}_{w \in W}$, and their faces $\{wC_I \cap S^{n-1}\}_{w \in W}$, where $I \subset S$, form a W-poset under inclusion, with W-action given by left translation. By Corollary A.2.3, this W-poset is isomorphic to the Coxeter poset X. By [26, Proposition 66.1], the simplicial complex X is W-isomorphic to the barycentric subdivision of the simplicial complex whose geometric realization is S^{n-1}. Hence there is a W equivariant homeomorphism $|X| \cong S^{n-1}$.

(ii) Since $|X| \cong S^{n-1}$. We know from the results on the homology of $(n-1)$-sphere that $H_i(C_\bullet(X)) = 0$ if $i \neq 0, n-1$, and $H_0(C_\bullet(X)) = H_{n-1}(C_\bullet(X)) \cong \mathbb{Q}$. From the definition of $H_0(C_\bullet(X))$ we know that it affords the trivial representation. The Lefschetz character $\mathcal{X}$ of $H_*(C_\bullet(X))$ is given by

$$\mathcal{X} = 1_W + (-1)^{n-1} \operatorname{Tr}(\cdot, H_{n-1}). \tag{A.2.1}$$

Now let s be any element in S. The fixed point $|X|^s \cong (S^{n-1})^s \cong S^{n-2}$, whose usual Euler characteristic is $1 + (-1)^{n-2}$. We know from Proposition A.1.6 that

$$\mathcal{X}(s) = 1 + (-1)^{n-2}.$$

Comparing with (A.2.1), we find $\operatorname{Tr}(s, H_{n-1}) = -1$ for all $s \in S$. Thus $H_{n-1}(C_\bullet(X))$ affords the sign representation.

$\square$

A.3 Proof of Howlett-Lehrer's Theorem for characters

Let $C_{I_0}(I)$ be defined as the set $\{w \in W \mid wI_0 \subset \langle I \rangle\}$.

Lemma A.3.1 (Howlett-Lehrer). *The set $R_{I_0} = \{wC_I \mid I \subset S,\ w \in C_{I_0}(I)\}$ is precisely the set of those W-regions contained in $I_0^\perp$.*

Corollary A.3.2. *The poset $X_{I_0^\perp} = \{wW_I \mid I \subset S,\ w \in C_{I_0}(I)\}$ (abbreviated as X_0) defines a subcomplex of the Coxeter complex X of W, which corresponds to the simplicial subdivision $\left\{S(I_0^\perp) \cap wC_I \mid I \subset S,\ w \in C_{I_0}(I)\right\}$ of $S(I_0^\perp)$.*

Theorem A.3.3 (R.B. Howlett, G.I. Lehrer)**.** *Let I_0 be a fixed subset of S, and let H be any subgroup of $N_W(W_{I_0})$. Let χ be a character of H, we have the following equation of characters:*

$$\sum_{I \subset S} (-1)^{|I|} \sum_{w \in W_I \backslash C_{I_0}(I)/H} \mathrm{Ind}_{H \cap W_I^w}^H (\mathrm{Res}_{H \cap W_I^w}^H(\chi)) = \widehat{\chi} := (-1)^{|I_0|}(-1)^{\ell_{I_0^\perp}(-)}\chi \tag{A.3.1}$$

where $I_0^\perp$ is its orthogonal complement of I_0.

We give now give a detailed proof of this theorem following [34].

Proof. We first prove for any subgroup $H \leqslant N_W(W_L)$,

$$\sum_{I \subset S} (-1)^{|I|} \sum_{w \in W_I \backslash C_{I_0}(I)/H} \mathrm{Ind}_{H \cap W_I^w}^H (\mathrm{Res}_{H \cap W_I^w}^H 1_H) = (-1)^{|I_0|}(-1)^{\ell_{I_0^\perp}(-)} 1_H. \tag{A.3.2}$$

Here the second summation is over a set of representatives for the (W_I, H)-double cosets contained in $C_{I_0}(I)$, and the notation 1_H denotes the character of trivial representation of H.

Let H be any subgroup of $N_W(W_{I_0})$, then H acts as a group of orthogonal transformation of $I_0^\perp$, hence as a group of homeomorphisms of the unit sphere $S(I_0^\perp)$ in $I_0^\perp$. Applying Theorem A.2.5, we obtain the Lefschetz number $\mathcal{X}_{I_0^\perp}$:

$$\mathcal{X}_{I_0^\perp}(h) = \sum_{i=0}^{\infty} (-1)^i \,\mathrm{Tr}(h, H_i(S(I_0^\perp))) = 1 + (-1)^{n-|I_0|-1} \det w|_{I_0^\perp}. \tag{A.3.3}$$

where $n = \dim V = \mathrm{Card}(S)$, and $\det w|_{I_0^\perp} = (-1)^{\ell_{I_0^\perp}(w)}$. From Proposition A.1.6, we have

$$\mathcal{X}_{I_0^\perp}(h) = \chi_{I_0^\perp}(X_0^h) = \sum_{i=0}^{\infty} (-1)^i \dim C_r(X_0^h).$$

From Definition A.2.4 and its relation (Corollary A.2.3) with the W-posets wC_I, we know X_0^h is the W-poset formed by

$$\{wW_I \mid I \subseteq \Delta,\ w \in C_{I_0}(I),\ hwW_I = wW_I\}.$$

The i-simplex $S(I_0^\perp \cap wC_I)$ in the simplicial subdivision of $S(I_0^\perp)$ corresponds to the i-chain of the following base

$$\{wW_I \mid I \subseteq \Delta,\ w \in C_{I_0}(I),\ hwW_I = wW_I,\ \mathrm{Card}(I) = n - 1 - i\}.$$

Thus we know

$$\mathcal{X}_{I_0^\perp}(h) = \sum_{I\subset S,\ I\neq S} (-1)^{n-1-|I|} n_{I_0,I}(h), \tag{A.3.4}$$

where $n_{I_0,I}(h)$ is the number of wW_I fixed by the left action of h in the subcomplex X_0, *i.e.*

$$\begin{aligned} n_{I_0,I}(h) &= \mathrm{Card}(\{xW_I \mid I \subseteq \Delta,\ x \in C_{I_0}(I),\ hxW_I = xW_I\}) \\ &= \sum_{w\in H\backslash C_{I_0}(I)/W_I} \mathrm{Card}(\{x \in HwW_I/W_I \mid hxW_I = xW_I\}) \\ &= \sum_{w\in H\backslash C_{I_0}(I)/W_I} \mathrm{Card}(\{x \in H/(H\cap {}^wW_I) \mid hx(H\cap {}^wW_I) = x(H\cap {}^wW_I)\}). \end{aligned} \tag{A.3.5}$$

The last equations follows from that the action of h on the set of cosets HwW_I/W_I is equivalent to the action of h on the set of cosets $H/(H\cap {}^wW_I)$. Recall that if G is any finite group and K is a subgroup of G, we define the operations of restriction and induction of class functions in the usual way. The induction to G of a class function ϕ of K is given

$$(\mathrm{Ind}_K^G \phi)(g) = \frac{1}{|K|}\sum_{x\in G} \phi\left(xgx^{-1}\right) = \phi(g)\frac{1}{|K|}\sum_{x\in G} 1 = \phi(g)(\mathrm{Ind}_K^G 1_K)(g), \tag{A.3.6}$$

summed over those elements $x \in G$ for which $xgx^{-1} \in K$. Especially, for $K = H \cap {}^wW_I$, $G = H$, we have

$$\mathrm{Card}(\{x \in H/(H\cap {}^wW_I) \mid hx(H\cap {}^wW_I) = x(H\cap {}^wW_I)\}) = (\mathrm{Ind}_{H\cap W_I^w}^H 1_{H\cap W_I^w})(h).$$

Thus from (A.3.5) we know

$$n_{I_0,I}(h) = \sum_{w\in W_I\backslash C_{I_0}(I)/H} (\mathrm{Ind}_{H\cap W_I^w}^H 1_{H\cap W_I^w})(h). \tag{A.3.7}$$

Substitute (A.3.7) into (A.3.4) and then compare (A.3.4) and (A.3.3), we have the following result

$$\sum_{I\subset S}(-1)^{|I|} \sum_{w\in W_I\backslash C_{I_0}(I)/H} \mathrm{Ind}_{H\cap W_I^w}^H(\mathrm{Res}_{H\cap W_I^w}^H 1_H) = (-1)^{|I_0|}(-1)^{\ell_{I_0^\perp}(-)} 1_H. \tag{A.3.8}$$

Using (A.3.6), we deduce (A.3.1) just by multiplying (A.3.8) ($H = W(\Lambda)$) by χ, an element in the character ring of H. □

Appendix B

Some examples of small rank groups

In this Appendix, we will apply Theorem 4.0.2 directly (without using knowledge of p-adic groups) to some affine Hecke algebras associated with small rank groups; notation follows [17, PLANCHE]. We will see that the involution maps the Steinberg representations and the trivial representations to each other.

B.1 Type $\widetilde{A_1}$

To avoid ambiguity, we do not use the abbreviations like T_μ for T_{t_μ}, or $q(x)$ for $q(t_x)$ in this subsection.

Consider the following based root datum:

$$\mathcal{R} = \left(X = \mathbb{Z},\ R = \{\pm 1\},\ Y = \mathbb{Z},\ R^\vee = \{\pm 2\},\ \Delta = \{1\}\right).$$

An affine Weyl group of this type is denoted by $W_{\widetilde{A_1}} = W(R, R^\vee)$, with the set $S_{\text{aff}} = \{s_\alpha,\ s_0 : x \mapsto 1 - x\}$. Denote by $\mathcal{H} = \mathcal{H}(W_{\widetilde{A_1}}, q_s)$ the affine Hecke algebra. We express the Bernstein-Lusztig basis in terms of T_{s_0} and T_α:

$$\theta_1 = q(t_1)^{-1/2} T_{t_1} = q_{s_0}^{-1/2} q_{s_\alpha}^{-1/2} T_{s_0} T_{s_\alpha},$$

and

$$\theta_n = q(t_1)^{-n/2} T_{t_1}^n. \tag{B.1.1}$$

We apply the involution appeared in Theorem 4.0.2 to θ_1:

$$\begin{aligned} \theta_1^* &= q_{s_0}^{-1/2} q_{s_\alpha}^{-1/2} q_{s_0} q_{s_\alpha} T_{t_{-1}}^{-1} = q(t_1)^{1/2} T_{t_{-1}}^{-1} = q_{s_0}^{1/2} q_{s_\alpha}^{1/2} T_{s_0}^{-1} T_{s_\alpha}^{-1} \\ &= \theta_1 - q_{s_0}^{-1/2}(q_{s_\alpha}^{1/2} - q_{s_\alpha}^{-1/2}) T_{s_0} - q_{s_\alpha}^{-1/2}(q_{s_0}^{1/2} - q_{s_0}^{-1/2}) T_{s_\alpha} + (q_{s_0}^{1/2} - q_{s_0}^{-1/2})(q_{s_\alpha}^{1/2} - q_{s_\alpha}^{-1/2}). \end{aligned} \tag{B.1.2}$$

We notice that $\theta_1^* \neq \theta_n$ for any n.

As is shown in [56], using Iwahori-Matsumoto presentation we can write the trivial representation and the Steinberg representation as $\mathbb{C}\{v_{\text{triv}}\}$ and $\mathbb{C}\{v_{\text{St}}\}$ such that the Hecke algebra action is as follows:

$$T_{s_\alpha}(v_{\text{triv}}) = q_{s_\alpha} v_{\text{triv}},\ T_{s_0}(v_{\text{triv}}) = q_{s_0} v_{\text{triv}}.$$

$$T_{s_\alpha}(v_{\text{St}}) = -v_{\text{St}},\ T_{s_0}(v_{\text{St}}) = -v_{\text{St}}.$$

We deduce that

$$\theta_1(v_{\text{triv}}) = q_{s_0}^{1/2} q_{s_\alpha}^{1/2} v_{\text{triv}},\ \theta_1(v_{\text{St}}) = q_{s_0}^{-1/2} q_{s_\alpha}^{-1/2} v_{\text{St}}.$$

We can compute the action twisted by the involution on v_{triv} and v_{St}:

$$T^*_{s_\alpha}(v_{\text{triv}}) = (-q_{s_\alpha})T^{-1}_{s_\alpha} v_{\text{triv}} = -v_{\text{triv}},\ \theta^*_1(v_{\text{triv}}) = q_{s_0}^{1/2} q_{s_\alpha}^{1/2} T^{-1}_{s_0} T^{-1}_{s_\alpha} v_{\text{triv}} = q_{s_0}^{-1/2} q_{s_\alpha}^{-1/2} v_{\text{triv}}.$$

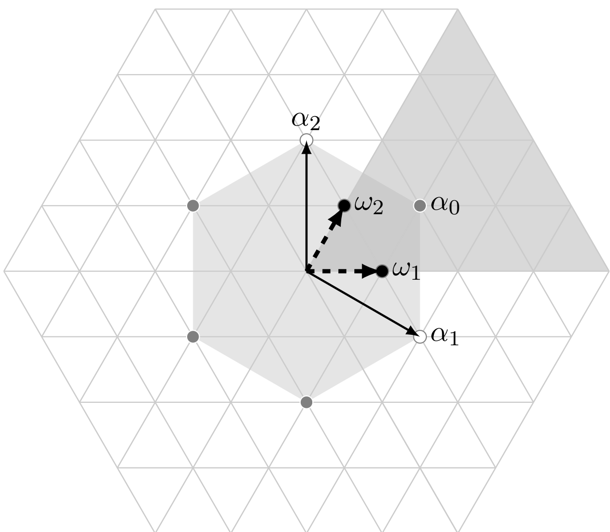
$$T^*_{s_\alpha}(v_{\text{St}}) = q_{s_\alpha} v_{\text{St}},\ \theta^*_1(v_{\text{St}}) = q_{s_0}^{1/2} q_{s_\alpha}^{1/2} v_{\text{St}}.$$

This verifies the special case of the theorem: we see after twisted by the involution, the action on v_{St} becomes the usual action on v_{Triv}. We conclude that

$$\mathrm{D}_{\mathcal{H}}(\mathbb{C}v_{\text{St}}) = (\mathbb{C}v_{\text{St}})^* = \mathbb{C}v_{\text{triv}}.$$

B.2 Type $\widetilde{A_2}$

Following [17], let V be the hyperplane of $\mathbb{R}^3 = \mathrm{Span}_{\mathbb{R}}(e_1, e_2, e_3)$ formed by points whose sum of coordinates is zero.



The simple roots are $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2 - e_3$, and their corresponding dominant weights are $\omega_1 = \frac{1}{3}(2\alpha_1+\alpha_2) = \frac{2}{3}e_1 - \frac{1}{3}e_2 - \frac{1}{3}e_3$, $\omega_2 = \frac{1}{3}(\alpha_1+2\alpha_2) = \frac{1}{3}e_1 + \frac{1}{3}e_2 - \frac{2}{3}e_3$. Let us denote the longest root by $\alpha_0 = \alpha_1+\alpha_2 = e_1 - e_3$.

Consider the following based root datum:

$$\mathcal{R} = \{X = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2,\ R = \{\pm\alpha_1,\ \pm\alpha_2,\ \pm\alpha_0\},\ X^\vee,\ R^\vee \cong R,\ \Delta = \{\alpha_1,\alpha_2\}\}.$$

An affine Weyl group of this type is denoted by $W_{\widetilde{A_2}} = W(\mathcal{R})$, and denote by

$$S_{\mathrm{aff}} = \{s_1, s_2, s_0 = s_1 s_2 s_1 t_{-\alpha_0}\}$$

where $s_i = s_{\alpha_i}$, $i = 1,2$, the set of simple reflections. Similar notation is also used for q-parameters and the elements in the Hecke algebra. Denote by $\mathcal{H} = \mathcal{H}(W_{\widetilde{A_2}}, q_s)$ the affine Hecke algebra.

Let us express the Bernstein-Lusztig basis in terms of T_i:

$$\begin{aligned}\theta_{\alpha_1} &= q(t_{2\alpha_1+\alpha_2})^{-1/2} q(t_{\alpha_1+\alpha_2})^{1/2} T_{2\alpha_1+\alpha_2} T^{-1}_{\alpha_1+\alpha_2} \\ &= q_0^{-1/2} q_2^{-1/2} T_0 T_2 T_1 T_0 T_1^{-1} T_0^{-1},\end{aligned} \tag{B.2.1}$$

and

$$\begin{aligned}\theta_{\alpha_0} &= q(t_{\alpha_1+\alpha_2})^{-1/2} T_{\alpha_1+\alpha_2} \\ &= q_0^{-1/2} q_2^{-1/2} q_1^{-1} T_0 T_1 T_2 T_1.\end{aligned} \tag{B.2.2}$$

We apply the involution appeared in Theorem 4.0.2 to θ_{α_1} and θ_{α_0}:

$$\theta^*_{\alpha_1} = q_0^{1/2} q_2^{1/2} T^{-1}_{-2\alpha_1-\alpha_2} T_{-\alpha_1-\alpha_2}, \tag{B.2.3}$$

and

$$\begin{aligned}\theta^*_{\alpha_0} &= q(t_{\alpha_1+\alpha_2})^{1/2} T^{-1}_{-\alpha_1-\alpha_2} \\ &= q_0^{1/2} q_2^{1/2} q_1 T_0^{-1} T_1^{-1} T_2^{-1} T_1^{-1}.\end{aligned} \tag{B.2.4}$$

We deduce that

$$\begin{aligned}&\theta^*_{\alpha_1}(v_{\mathrm{triv}}) = q_0^{-1/2} q_1^{-1/2}, \qquad \theta^*_{\alpha_1}(v_{\mathrm{St}}) = q_0^{1/2} q_2^{1/2}, \quad \theta^*_{\alpha_0}(v_{\mathrm{triv}}) = q_0^{-1/2} q_1^{-1} q_2^{-1/2} v_{\mathrm{triv}}, \\ &\theta^*_{\alpha_0}(v_{\mathrm{St}}) = q_0^{1/2} q_2^{1/2} q_1 v_{\mathrm{St}}, \quad T_1^*(v_{\mathrm{triv}}) = -v_{\mathrm{triv}}, \qquad T_1^*(v_{\mathrm{St}}) = q_1 v_{\mathrm{St}}, \\ &T_2^*(v_{\mathrm{triv}}) = -v_{\mathrm{triv}}, \qquad T_2^*(v_{\mathrm{St}}) = q_2 v_{\mathrm{St}}.\end{aligned}$$

And we also have

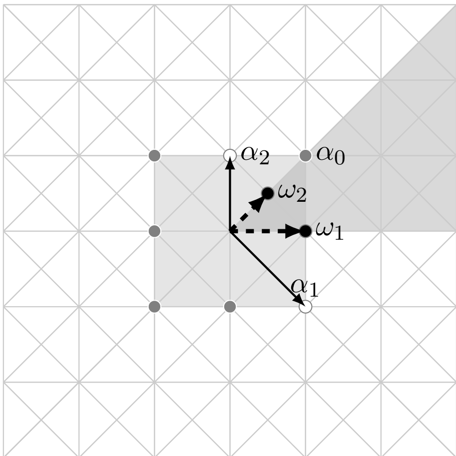
$$\begin{aligned}&\theta^*_{\alpha_1}(v_{\mathrm{triv}}) = q_0^{-1/2} q_1^{-1/2}, \qquad \theta^*_{\alpha_1}(v_{\mathrm{St}}) = q_0^{1/2} q_2^{1/2}, \quad \theta^*_{\alpha_0}(v_{\mathrm{triv}}) = q_0^{-1/2} q_1^{-1} q_2^{-1/2} v_{\mathrm{triv}}, \\ &\theta^*_{\alpha_0}(v_{\mathrm{St}}) = q_0^{1/2} q_2^{1/2} q_1 v_{\mathrm{St}}, \quad T_1^*(v_{\mathrm{triv}}) = -v_{\mathrm{triv}}, \qquad T_1^*(v_{\mathrm{St}}) = q_1 v_{\mathrm{St}}, \\ &T_2^*(v_{\mathrm{triv}}) = -v_{\mathrm{triv}}, \qquad T_2^*(v_{\mathrm{St}}) = q_2 v_{\mathrm{St}}.\end{aligned}$$

This verifies the special case of the theorem: we see after twisted by the involution, the action on v_{St} becomes the usual action on v_{Triv}. We conclude that

$$\mathrm{D}_{\mathcal{H}}(\mathbb{C}v_{\mathrm{St}}) = (\mathbb{C}v_{\mathrm{St}})^* = \mathbb{C}v_{\mathrm{triv}}.$$

B.3 Type $\widetilde{B_2}$

The definition of affine Weyl groups used by [17] is different from ours, the group generated by reflections with respect to affine hyperplanes in the picture below is $\mathbb{Z}R^\vee \rtimes W$. Let V be $\mathbb{R}^2 = \mathrm{Span}_{\mathbb{R}}(e_1, e_2)$.



The simple roots are $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2$. The corresponding dominant weights are $\omega_1 = \alpha_1+\alpha_2 = e_1$, $\omega_2 = \dfrac{1}{2}(\alpha_1+2\alpha_2) = \dfrac{e_1+e_2}{2}$. The longest root is $\alpha_0 = \alpha_1+2\alpha_2 = e_1+e_2$.

Consider the following based root datum:

$$\begin{aligned}\mathcal{R} = \{X = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2,\ R = \{\pm\alpha_1,\ \pm\alpha_2,\ \pm\alpha_0,\ \pm\omega_1\}, Y = X^\vee, \\ R^\vee = \{\pm\alpha_1,\ \pm 2\alpha_2,\ \pm\alpha_0,\ \pm 2\omega_1\}, \Delta = \{\alpha_1, \alpha_2\}\}.\end{aligned} \tag{B.3.1}$$

An affine Weyl group of this type is denoted by $W_{\widetilde{B_2}} = W(\mathcal{R}^\vee)$ with the set $S_{\mathrm{aff}} = \{s_1, s_2, s_0 = t_{\alpha_0}s_2s_1s_2\}$ where $s_i = s_{\alpha_i}$, $i = 1, 2$ for simplification. Similar notation is also used for q-parameters and the elements in the Hecke algebra. Denote by $\mathcal{H} = \mathcal{H}(W_{\widetilde{B_2}}, q_s)$ the affine Hecke algebra.

Let us express the Bernstein-Lusztig basis in terms of T_i:

$$\theta_{2\omega_1} = q(t_{2\omega_1})^{-1/2}T_{2\omega_1} = q_0^{-1}q_2^{-1}q_1^{-1}(T_0T_2T_1)^2,$$

and

$$\theta_{\alpha_0} = q(t_{\alpha_0})^{-1/2}T_{\alpha_0} = q_0^{-1/2}q_2^{-1}q_1^{-1/2}T_0T_2T_1T_2.$$

We apply the involution appeared in Theorem 4.0.2 to $\theta_{2\omega_1}$ and θ_{α_0}:

$$\theta^*_{2\omega_1} = q(t_{2\omega_1})^{1/2} T^{-1}_{-2\omega_1} = q_0 q_2 q_1 (T_1 T_2 T_0)^{-2},$$

and

$$\theta^*_{\alpha_0} = q(t_{\alpha_0})^{1/2} T^{-1}_{-\alpha_0} = q_0^{1/2} q_2 q_1^{1/2} T_0^{-1} T_2^{-1} T_1^{-1} T_2^{-1}.$$

We deduce that

$$\theta_{2\omega_1}(v_{\text{triv}}) = q_0 q_1 q_2 v_{\text{triv}},\ \ \theta_{2\omega_1}(v_{\text{St}}) = q_0^{-1} q_1^{-1} q_2^{-1} v_{\text{St}},$$
$$\theta_{\alpha_0}(v_{\text{triv}}) = q_0^{1/2} q_1^{1/2} q_2 v_{\text{triv}}, \theta_{\alpha_0}(v_{\text{St}}) = q_0^{-1/2} q_1^{-1/2} q_2^{-1} v_{\text{St}}.$$

And we also have

$$\theta^*_{2\omega_1}(v_{\text{triv}}) = q_0^{-1} q_1^{-1} q_2^{-1} v_{\text{triv}},\ \ \theta^*_{2\omega_1}(v_{\text{St}}) = q_0 q_1 q_2 v_{\text{St}},$$
$$\theta^*_{\alpha_0}(v_{\text{triv}}) = q_0^{-1/2} q_1^{-1/2} q_2^{-1} v_{\text{triv}},\ \ \theta^*_{\alpha_0}(v_{\text{St}}) = q_0^{1/2} q_1^{1/2} q_2 v_{\text{St}}.$$

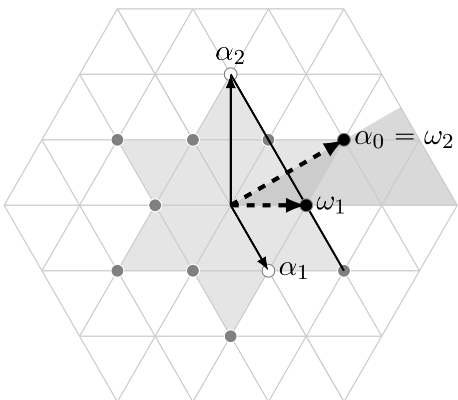
Computations for the finite Hecke algebra part are the same as the previous example.

This verifies the special case of the theorem: we see that after twisted by the involution, the action on v_{St} becomes the usual action on v_{Triv}. We conclude that

$$\mathrm{D}_{\mathcal{H}}(\mathbb{C}v_{\text{St}}) = (\mathbb{C}v_{\text{St}})^* = \mathbb{C}v_{\text{triv}}.$$

B.4 Type $\widetilde{G_2}$

Let V be the hyperplane of $\mathbb{R}^3 = \mathrm{Span}_{\mathbb{R}}(e_1, e_2, e_3)$ formed by points whose sum of coordinates is zero.



The simple roots are $\alpha_1 = \dfrac{1}{\sqrt{3}}(e_1 - e_2)$, $\alpha_2 = \dfrac{1}{\sqrt{3}}(-2e_1 + e_2 + e_3)$. The corresponding dominant weights are $\omega_1 = 2\alpha_1 + \alpha_2 = \dfrac{1}{\sqrt{3}}(-e_2 + e_3)$, $\omega_2 = 3\alpha_1 + 2\alpha_2 = \dfrac{1}{\sqrt{3}}(-e_1 - e_2 + 2e_3)$. The longest root is $\alpha_0 = \omega_2$.

Consider the following based root datum:

$$\begin{aligned}\mathcal{R} = \{&X = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2,\ R = \{\pm\alpha_1,\ \pm\alpha_2,\ \pm(\alpha_1+\alpha_2),\ \pm\omega_1,\ \pm(3\alpha_1+\alpha_2),\ \pm\alpha_0\},\\ &Y = 3\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2,\ R^\vee = \{\pm 3\alpha_1,\ \pm\alpha_2,\ \pm 3(\alpha_1+\alpha_2),\ \pm 3\omega_1,\ \pm(3\alpha_1+\alpha_2),\ \pm\alpha_0\},\\ &\Delta = \{\alpha_1,\alpha_2\}\}.\end{aligned} \tag{B.4.1}$$

An affine Weyl group of this type is denoted by $W_{\widetilde{G_2}} = W(\mathcal{R}^\vee)$ with the set $S_{\mathrm{aff}} = \{s_1, s_2, s_0 = t_{\alpha_0}s_2s_1s_2s_1s_2\}$ where $s_i = s_{\alpha_i},\ i = 1,2$ for simplification. Similar notation is also used for q-parameters and the elements in the Hecke algebra. Denote by $\mathcal{H} = \mathcal{H}(W_{\widetilde{G_2}}, q_s)$ the affine Hecke algebra.

Let us express the Bernstein-Lusztig basis in terms of T_i:

$$\theta_{3\omega_1} = q^{-1/2}(t_{3\omega_1})T_{3\omega_1} = q_0^{-1}q_1^{-2}q_2^{-2}(T_0T_2T_1T_2T_1)^2,$$

and

$$\theta_{\omega_2} = q^{-1/2}(t_{\omega_2})T_{\omega_2} = q_0^{-1/2}q_1^{-1}q_2^{-3/2}T_0T_2T_1T_2T_1T_2.$$

We apply the involution appeared in Theorem 4.0.2 to $\theta_{3\omega_1}$ and θ_{ω_2}:

$$\theta^*_{3\omega_1} = q^{1/2}(t_{3\omega_1})T^{-1}_{-3\omega_1} = q_0q_1^2q_2^2(T_1T_2T_1T_2T_0)^{-2},$$

and

$$\theta^*_{\omega_2} = q^{1/2}(t_{\omega_2})T^{-1}_{-\omega_2} = q_0^{1/2}q_1q_2^{3/2}T_0^{-1}T_2^{-1}T_1^{-1}T_2^{-1}T_1^{-1}T_2^{-1}.$$

We deduce that

$$\begin{aligned}\theta_{3\omega_1}(v_{\mathrm{triv}}) &= q_0q_1^2q_2^2v_{\mathrm{triv}}, & \theta_{3\omega_1}(v_{\mathrm{St}}) &= q_0^{-1}q_1^{-2}q_2^{-2}v_{\mathrm{St}},\\ \theta_{\omega_2}(v_{\mathrm{triv}}) &= q_0^{1/2}q_1q_2^{3/2}v_{\mathrm{triv}}, & \theta_{\omega_2}(v_{\mathrm{St}}) &= q_0^{-1/2}q_1^{-1}q_2^{-3/2}v_{\mathrm{St}}.\end{aligned}$$

And we also have

$$\begin{aligned}\theta^*_{3\omega_1}(v_{\mathrm{triv}}) &= q_0^{-1}q_1^{-2}q_2^{-2}v_{\mathrm{triv}}, & \theta^*_{3\omega_1}(v_{\mathrm{St}}) &= q_0q_1^2q_2^2v_{\mathrm{St}},\\ \theta^*_{\omega_2}(v_{\mathrm{triv}}) &= q_0^{-1/2}q_1^{-1}q_2^{-3/2}v_{\mathrm{triv}}, & \theta^*_{\omega_2}(v_{\mathrm{St}}) &= q_0^{1/2}q_1q_2^{3/2}v_{\mathrm{St}}.\end{aligned}$$

This verifies the special case of the theorem: we see that after twisted by the involution, the action on v_{St} becomes the usual action on v_{Triv}. We conclude that

$$\mathrm{D}_{\mathcal{H}}(\mathbb{C}v_{\mathrm{St}}) = (\mathbb{C}v_{\mathrm{St}})^* = \mathbb{C}v_{\mathrm{triv}}.$$

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