1. Solution to Exercise 10 in Assignment 4

Solution. By Theorem 5.4.1 of [1], we have

$$\operatorname{Im}(A)^\perp = \operatorname{Ker}(A^T)$$

Therefore we have (by taking orthogonal complement)

$$\operatorname{Im}(A) = \operatorname{Ker}(A^T)^\perp \overset{\text{Thm 5.4.2 of [1]}}{=\!=\!=\!=} \operatorname{Ker}(AA^T)^\perp \overset{\text{Thm 5.4.1 of [1]}}{=\!=\!=\!=} \operatorname{Im}(AA^T).$$

Or we can argue by rank nullity theorem:

Solution. We see easily that $\operatorname{Ker} A^T \subset \operatorname{Ker} AA^T$. On the other hand, let $x \in \operatorname{Ker}(AA^T)$, we have

$$\begin{aligned} AA^Tx = 0 \text{ (here 0 is a vector)} &\Rightarrow x^T AA^T x = 0 \\ &\Rightarrow (A^Tx)^T(A^Tx) = 0 \Rightarrow ||A^Tx||^2 = 0 \Rightarrow A^Tx = 0 \\ &\Rightarrow x \in \operatorname{Ker} A^T \Rightarrow \operatorname{Ker}(AA^T) \subset \operatorname{Ker} A^T \end{aligned} \tag{1.1}$$

Therefore we have $\operatorname{Ker} AA^T = \operatorname{Ker} A^T$ $(*)$ (or it follows from Thm 5.4.2 of [1]). Applying rank nullity theorem to A^T, we have

$$n = \dim \operatorname{Ker} A^T + \operatorname{rank} A^T.$$

Applying rank nullity theorem to AA^T, we have

$$n = \dim \operatorname{Ker}(AA^T) + \operatorname{rank}(AA^T)$$

From $(*)$, we know $\dim \operatorname{Ker} A^T = \dim \operatorname{Ker}(AA^T)$, and we have $\dim \operatorname{Im}(A) = \operatorname{rank} A = \operatorname{rank} A^T = \operatorname{rank}(AA^T) = \dim \operatorname{Im}(AA^T)$. We can easily prove $\operatorname{Im}(AA^T) \subset \operatorname{Im}(A)$. Therefore we have $\operatorname{Im}(AA^T) = \operatorname{Im}(A)$.

Remark. The statement is false for complex coefficients matrix, for example, $\begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$ where $i^2 = -1$.

References

[1] Otto Bretscher, *Linear Algebra with Application*, 5th ed., Pearson, December 20, 2012.