# SOME NOTES ON LOCAL LANGLANDS CORRESPONDENCE

Contents



## 1. Weil Groups

For this section, reference is [1] and the article *number theoretic background* by J.Tate in [2].

The language of class formation is axiomatic approach to handle local and global class field theory. For example, when $K$ is a finite algebraic number field, the formation module $A$ can be $K^\times$, idèle group of $K$ or idèle class groups of $K$.

Let $G$ be a topological group,

**Definition 1.1.** A *formation* $(G, \{G_F\}; A)$ consists of:

(1) A group $G$, together with an indexed family $\{G_F\}_{F\in\Sigma}$ of subgroups of $G$ satisfying the following conditions:
   (a) Each element of $\{G_F\}$ is of finite index in $G$.
   (b) Each subgroup of $G$ which contains a member of the family $\{G_F\}$ also belongs to the family.
   (c) The intersection of two members of the family $\{G_F\}$ also belongs to the family.
   (d) Any conjugate of a member of the family $\{G_F\}$ is also a member of the family.

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(e) The intersection of all members of the family $\{G_F\}$ is the identity:

$$\bigcap_{F\in\Sigma} G_F = 1$$

(2) A $G$-module $A$ such that $A = \bigcup_{F\in\Sigma} A^{G_F}$, in other words, such that every element of $A$ is left fixed by some member of the family $\{G_F\}$.

We call the submodule $A_F := A^{G_F}$ in (2) above the $F$-*level*. The index $(G_F : G_K)$ which is finite by assumption is called the degree of the layer $K/F$ and is denoted by $[K : F]$. The layer is called a *normal layer* if $G_K$ is a normal subgroup of $G_F$. The factor group $G_F/G_K$ is called the *galois group* of the normal layer. Fix notation: $H^r(K/F) := H^r(G_F/G_K, A_K)$, and $H^2(*/F) := \varinjlim_K H^2(K/F)$ where $K/F$ normal.

If moreover, the following axioms are satisfied, then the formation is called a *class formation.*

**Axiom 0**: In each cyclic layer of prime degree, the Herbrand quotient $h_{2/1}$ is defined and equal to the degree.

**Axiom I**: (*Field Formation Axiom*) $H^1(K/F) = 0$ for all normal layer $K/F$.

**Axiom II**: For each field $F$, there is an isomorphism $\alpha \to \mathrm{Inv}_F\,\alpha$ of the Brauer group into $\mathbb{Q}/\mathbb{Z}$, such that:

(a) If $K/F$ is a normal layer of degree $n$, then image of $H^2(K/F)$ is $\frac{1}{n}\mathbb{Z}/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$.
(b) For each layer $E/F$ of degree $n$ we have

$$\mathrm{Inv}_E \,\mathrm{Res}_{F,E} = n\,\mathrm{Inv}_F$$

Let us assume $(G, \{G_F\}, A)$ is a class formation, $H^2(K/F)$ is isomorphic to $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$, any rational number $t$ which can be written with denominator $n$ determines a unique $\alpha \in H^2(K/F)$ such that $\mathrm{Inv}_F\,\alpha \equiv t \pmod{\mathbb{Z}}$, this $\alpha$ is called the cohomology class with *invariant* $t$. If we are working with a complex $X$ for the Galois group $G_{K/F}$ of the layer, and $f : X_2 \to A_K$ is a cocycle in the class $\alpha$, call $f$ is a cocycle with invariant $t$. The class with invariant $1/n$ generates $H^2(K/F)$, it is called *fundamental class* of layer $K/F$, cocycle $f$ representing it is called a fundamental 2-cocycle.

**Definition 1.2. (Weil Group for a normal layer)** Let $K/F$ be a normal layer in a class formation. A *Weil group* $(U, g, \{f_E\})$ for the layer $K/F$ consists of the following objects:

(1) A group $U$.
(2) A homomorphism $g$ of $U$ onto the Galois group $G_{K/F}$. And define for each intermediate field $F \subset E \subset K$, the subgroup $U_E = g^{-1}(G_{K/E})$.
(3) A set of homomorphisms $f_E : A_E \cong U_E/U_E^c$ of the $E$-level onto the factor commutator group of $U_E$, one for each intermediate field.

such that $(U, g, \{f_E\})$ satisfying:

(a) For each intermediate layer $E'/E$, $F \subset E \subset E' \subset K$, the following diagram is commutative:

$$\begin{array}{ccc} A_E & \xrightarrow{\cong} & U_E/U_E^c \\ \big\downarrow & & \big\downarrow V_{E'/E} \\ A_{E'} & \xrightarrow{\cong} & U_{E'}/U_{E'}^c \end{array}$$

where horizontal isomorphisms are induced by $f_E$, $f_{E'}$, left vertical is inclusion map and right vertical arrow is the group theoretic transfer (Verlagerung, see [1] Chapter XIII or Serre Chapter VII) from $U_E$ to $U_{E'}$.

(b) Let $u$ be an element of $U$ and put $\sigma = g(u) \in G_{K/F}$. Then it is clear that $U_E^u = U_{E^\sigma}$. Then the following diagram is commutative:

$$\begin{array}{ccc} A_E & \xrightarrow{\cong} & U_E/U_E^c \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle u} \\ A_{E^\sigma} & \xrightarrow{\cong} & U_{E^\sigma}/U_{E^\sigma}^c \end{array}$$

where the right vertical arrow is the map of the factor commutator groups induced by conjugation by $u : U_E \to uU_Eu^{-1} = U_{E^\sigma}$.

(c) Suppose $L/E$ is a normal intermediate level, $F \subset E \subset L \subset K$. Then the map $g$ induces an isomorphism

$$U_E/U_L \cong G_{K/E}/G_{K/L} = G_{L/E}$$

Since $f_L : A_L \xrightarrow{\cong} U_L/U_L^c$, $U_E/U_L^c$ can be viewed as a group extension of $A_L$ by $G_{L/E}$ as follows:

$$1 \longrightarrow A_L \cong U_L/U_L^c \longrightarrow U_E/U_L^c \longrightarrow U_E/U_L \cong G_{L/E} \longrightarrow 1 \tag{1}$$

The operation of $G_{L/E}$ on $A_L$ associated with this extension is the natural one. Property $(c)$ requires that the class of extension in 1 is the fundamental class $\alpha_{L/E}$ of the layer $L/E$.

(d) $U_K^c = 1$

**Theorem 1.1.** ***(Existence of Weil Group for normal layers))*** *Let $K/F$ be a normal layer in a class formation. Then there exists a Weil group $(U, g, \{f_E\})$ for the layer $K/F$.*

We can see that a Weil group for a big normal layer $K_1/F_1$ contains information about all intermediate layers $K/F$ (see [1] Chapter XV Theorem 3). This suggests the definition of Weil group for the whole class formation:

**Definition 1.3.** Let $(G, \{G_F\}, A)$ be a topological class formation. A Weil group $(U, g, \{f_F\})$ for the formation consists of the following objects:

(1) A topological group $U$.
(2) A representation $g$ of $U$ onto the dense subgroup of the Galois group $G$ of the formation.
(3) For each $F$ of our formation, an isomorphism

$$f_F : A_F \cong U_F/U_F^c$$

where $U_F^c$ denotes the closure of the commutator subgroup $U_F$.

In order to constitute a Weil group, $(U, g, \{f_F\})$ must have the following properties:

(a) For each layer $E/F$, then the following diagram commutes:

$$\begin{array}{ccc} A_F & \xrightarrow{\cong} & U_F/U_F^c \\ \downarrow & & \downarrow{\scriptstyle V} \\ A_E & \xrightarrow{\cong} & U_E/U_E^c \end{array}$$

where $V$ is the transfer map.

(b) Let $u \in U$ and let $\sigma = g(u) \in G$. Then it is clear that $u(U_E)u^{-1} = U_{E^\sigma}$, then the following diagram commutes for each $E$:

$$\begin{array}{ccc} A_E & \xrightarrow{\cong} & U_E/U_E^c \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle u} \\ A_{E^\sigma} & \xrightarrow{\cong} & U_{E^\sigma}/U_{E^\sigma}^c \end{array}$$

(c) For each normal layer $K/F$, the class of the group extension

$$1 \longrightarrow A_K \cong U_K/U_K^c \longrightarrow U_F/U_K^c \longrightarrow U_F/U_K \cong G_{K/F} \longrightarrow 1 \tag{2}$$

is the fundamental class of the layer $K/F$.

(d) We finally requires that
$$U \to \varprojlim U/U_K^c$$
is an isomorphism of topological groups.

If $k$ is the ground field, then $U/U_K^c$ for variable $K$ normal over $k$ is the Weil group for the normal layer $K/k$.

For proofs of the following two theorems, please see [1, Artin-Tate] Chapter XV, theorem 7 and theorem 8.

**Theorem 1.2.** *Suppose $(G, \{G_F\}, A)$ is a topological class formation satisfying the following three conditions:*

*(a) The norm map $N_{E/F} : A_E \to A_F$ is an open map for each layer $E/F$.*

*(b) The factor group $A_E/A_F$ is compact for each layer $E/F$.*

*(c) The Galois group $G$ is complete.*

*Then there exists a Weil group $(U, g, \{f_F\})$ for the formation, and it is unique up to isomorphism.*

**Theorem 1.3.** *Let $(U, g, \{f_F\})$ be a Weil group for a class formation $(U, G_F, A)$. For each field $F$, the composed map*

$$A_F \xrightarrow{f_F} U_F^{ab} \xrightarrow{g_F^{ab}} G_F^{ab} \tag{3}$$

*is the reciprocity map, where $g_F^{ab}$ is induced by $g$.*

*Moreover, if every normal layer $K/k$ there is a cyclic $L/k$ of the same degree, then in the definition of Weil group for a class formation, we can substitute the above condition for (c) of definition 1.3.*

## 2. Local Langlands Correspondence for torus

This section mainly follows the original paper of R.P.Langlands [7] and paper by J.P.Labesse [8]. The structure (dividing proof into three parts, and preparations) follows the relevant materials in [5]. And the article [6] gives me some help for understanding some details in the original paper.

### 2.1. Some Preparations.

2.1.1. *Some definitions.* Let $K$ be an algebraic number field, let $S_K$ denote the set of prime divisors, $S_\infty$ the set of infinite prime divisors. Assume $S$ is a finite subset containing $S_\infty$, we define:

$$\mathbb{A}_{K,S} = \prod_{v\in S} K_v \times \prod_{v\notin S} O_v$$

and give $\mathbb{A}_{K,S}$ the product topology. We call $\mathbb{A}_{K,S}$ the *ring of $S$-adèles.*

And define *the ring of adèles* of $K$ to be: $\mathbb{A}_K := \varinjlim_S \mathbb{A}_{K,S}$.

We know (see [11] Chapter II and Chapter VIII) the following facts: $\mathbb{A}_K$ is locally compact; $\mathbb{A}_K(S)$ are all open subsets of $\mathbb{A}_K$; if $L/K$ is a finite Galois extension, then $\mathbb{A}_L \cong \mathbb{A}_K \otimes_K L$.

Now we define idèle group of a global field $K$:

Let $S$ be as above, define *group of $S$-idèles*:

$$J_{K,S} := \prod_{v \in S} K_v^\times \times \prod_{v \notin S} U_v$$

where $U_v$ is group of units in $K_v$, and give it the product topology.

The *idèle group* $J_K$ is defined to be $J_K := \varinjlim_S J_{K,S}$. $K^\times$ is discrete in $J_K$.

We define *idèle class group of* $C_K$ to be $J_K/K^\times$.

2.1.2. *Complements on group cohomology.* We have known terminology for finite group cohomology from [10] chapter VII. Since we have defined class formation, we add a few more terms:

Let $K/F$ be a Galois extension of field $F$, $G = \mathrm{Gal}(K/F)$, $\{E\}$ denotes the set of all finite galois extensions of $F$, $G_E$ is the subgroup of $G$ fixing $E$. $((G, \{G_E\}, K)$ is a formation.)

The action of $G$ on $K^\times$ makes it into a $G$-module, since $(K^\times)^{G_E} = E^\times$, we know $K^\times = \bigcup E^\times$.($K^\times$ is discrete $G$-module) Further, $E^\times$ is $G/G_E$-module, we have:

$$H^n_{ct}(G, K^\times) \cong \varinjlim_E H^n(G/G_E, E^\times)$$

where $H^n_{ct}$ is defined using continuous cocycles.

We also have *Hilbert 90* for our case:

**Theorem 2.1.**

$$H^1_{ct}(G, K^\times) = 0$$

The proof is similar to finite case, then pass to limit.

2.1.3. *A Commutative diagram.* In the following of this subsection, we fix an exact sequence:

$$1 \longrightarrow C \xrightarrow{\ i\ } W \xrightarrow{\ j\ } G \longrightarrow 1$$

where $C$ is normal subgroup of $W$, and any $G$-module is viewed as a $W$-module through $j$, also a $C$-module with trivial $C$-action.

For 1-cycle $x : a \mapsto x(a)$ of $C$ on $A$, we define the corresponding 1-cycle of $W$ on $A$ by trivial extension. For 1-cycle of $W$ on $A$ $x : w \mapsto x(w)$, we define the corresponding 1-cycle on $G$:

$$\sigma \mapsto \sum_{j(w)=\sigma} x(w)$$

By explicitly computation of cycles, we have:

**Proposition 2.2.** *The following sequence is exact.*

$$H_1(C, A) \longrightarrow H_1(W, A) \longrightarrow H_1(G, A) \longrightarrow 0 \tag{4}$$

There is an important result concerning the composition of Cor : $H_1(C, A) \to H_1(W, A)$ and Res : $H_1(W, A) \to H_1(C, A)$, and the $N_G$ map.

**Proposition 2.3.** *The following diagram commutes:*

$$\begin{array}{ccc} H_1(C,A) & \xrightarrow{\text{Cor}} & H_1(W,A) \\ \downarrow N_G & & \downarrow \text{Res} \\ N_G(H_1(C,A)) & \longrightarrow & H_1(C,A) \end{array} \tag{5}$$

*Proof.* From the exact sequence:

$$0 \longrightarrow I_G \longrightarrow \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0 \tag{6}$$

where $\epsilon : \mathbb{Z}[g] \to \mathbb{Z}$ defined by $\sum n_\sigma \sigma \mapsto \sum n_\sigma$. We tensor it with a $\mathbb{Z}$-free $G$-module, and using homological sequence, we have:

$$H_1(G,A) \xrightarrow{\sim} H_0(G, I_G \otimes A) = I_G \otimes A$$

For $G$ replaced by $W$, we have $H_1(W,A) \xrightarrow{\sim} H_0(W, I_w \otimes A)$, similarly for $C$.

We will mainly work with the following diagram:

$$\begin{array}{ccccccc} & & H_1(C,A) & \xrightarrow{\text{Cor}} & H_1(W,A) & \xrightarrow{\sim} & H_0(W, I_W \otimes A)) \\ & & \downarrow N_G & & \downarrow \text{Res} & & \downarrow \text{Res} \\ 0 & \longrightarrow & N_G(H_1(C,A)) & \hookrightarrow & H_1(C,A) & \xrightarrow{\sim} & H_0(C, I_W \otimes A) \end{array}$$

We already know that the right square is commutative,

If $x : w \mapsto x(w)$, $x$ is 1-cycle of $W$ in $A$, then its image in $H_0(W, I_W \otimes A)$ is $\sum_w (w^{-1} - 1)(1 \otimes x(w))$, its restriction to $C$ is:

$$\sum_\sigma \sum_w (w_\sigma w^{-1}(1 \otimes x(w)) - w_\sigma(1 \otimes x(w)))$$

We have relation: $w_\tau w = c_{\tau,w} w_\sigma$, then above sum is:

$$\sum_\tau \sum_w (c_{\tau,w}^{-1} - 1) w_t (1 \otimes x(w))$$

which equals to:

$$\sum_{c \in C} \left( (c^{-1} - 1) \sum_{c_{\tau,w} = c} 1 \otimes w_\tau x(w) \right)$$

this is homological class of the following 1-cycle in $H_1(C,A)$:

$$y : c \mapsto \sum_{c_{\tau,w} = c} w_\tau x(w)$$

If support of $x$ is in $C$, then:

$$y(c) = \sum_{w_\tau b w_\tau^{-1} = c} w_\tau x(b) = \sum_\tau \tau x(\tau^{-1}(c)) = \sum_\tau {}^\tau x(c)$$

So diagram 5 commutes. □

2.1.4. *Cup Product.* There are some dual theorems due to Tate and Nakayama (see [10] IX section 8 and XI ANNEXE), of most interest to us is the explicit calculations of cup product:

From now on till the end of this document, $\widehat{H}$ is used to denote the Tate cohomological groups.

**Proposition 2.4.** *Let $A$, $B$ be $G$-modules.*

*(1) For $a \in A^G$, let $f_a : B \to A \otimes B$ be $G$-morphism given by $b \mapsto a \otimes b$, then cup product:*

$$\widehat{H}^0(G,A) \otimes \widehat{H}^n(G,B) \to \widehat{H}^n(G,A \otimes B)$$

*is given by:*

$$[a] \cup [x] = f_a^*([x])$$

*where $[a]$ denotes the class of $a$, $[x] \in \widehat{H}^n(G,B)$.*

*(2) Cup product:*

$$\widehat{H}^1(G,A) \otimes \widehat{H}^{-1}(G,B) \to \widehat{H}^0(G,A \otimes B)$$

*is induced by:*

$$[f] \cup [b] = [\sum_{\sigma \in G} f(\sigma) \otimes \sigma b]$$

*where $b \in B$ satisfies $N_G b = 0$, $f$ is 1-cocycle.*

*(3) Cup product:*

$$\widehat{H}^1(G,A) \otimes \widehat{H}^{-2}(G,B) \to \widehat{H}^{-1}(G,A \otimes B)$$

*is induced by:*

$$[f] \cup [x] = [\sum_{\sigma \in G} f(\sigma) \otimes x(\sigma)]$$

*(4) Cup product:*

$$\widehat{H}^{-2}(G,A) \otimes \widehat{H}^2(G,B) \to \widehat{H}^0(G,A \otimes B)$$

*is induced by:*

$$[x] \cup [f] = [\sum_{\sigma,\tau \in G} \tau x(\sigma) \otimes f(\tau,\sigma)]$$

Assume $A$ is a free $G$-module, $Q$ is a trivial $G$-module, then the above (3) gives a pairing:

$$H^1(G, \mathrm{Hom}(A,Q)) \times H_1(G,A) \to H_0(G,Q) = Q$$

therefore we have a morphism:

$$\Phi : H^1(G, \mathrm{Hom}(A,Q)) \to \mathrm{Hom}(H_1(G,A),Q) \tag{7}$$

**Proposition 2.5.** *If $Q$ is $\mathbb{Z}$-injective, then $\Phi$ above is an isomorphism.*

*Proof.* See [7] p11-12 or [5] part 3 proposition 1.3.8.. □

**Proposition 2.6.** *If $G$ is a finite group, $C$ is a class module, $u \in H^2(G,C)$ is a fundamental class,*

$$1 \longrightarrow C \xrightarrow{i} W \xrightarrow{j} G \longrightarrow 1$$

*is a group extension belongs to class $u$. Assume $A$ is a $\mathbb{Z}$-free $G$-module, $Z = \mathrm{Ker}\,(N_G : H_1(C,A) \to H_1(C,A))$, then the following is exact:*

$$0 \longrightarrow Z \longrightarrow H_1(C,A) \longrightarrow H_1(W,A) \longrightarrow H_1(G,A) \longrightarrow 0 \tag{8}$$

*Proof.* See [7] p12-13 or [5] part 3 proposition 1.3.8.. □

2.1.5. *Galois Cohomological Groups of multiplication groups and unit groups of local fields.* In this subsection, let us use $F$ to denote the completion of a number field at a finite place $v$, $\overline{F}$ is the algebraic closure of $F$. We use $O_F$, $U_F$, $P_F$ to denote $O_v$, $U_v$, $P_v$. We first check the axioms of class formation are satisfied.

Let $K/F$ be Galois extension of degree $n$, $G = G_{K/F}$, assume $H$ is subgroup of $G$ of order $m$, from Hilbert 90 we have $H^1(G, K^\times) = 0$. Assume $F'$ is invariant field of $H$, then $H = G_{K/F'}$. Then $H^2(H, K^\times)$ is cyclic group of order $m$ generated by $u_{K/F}$. By calculations:

$$\mathrm{Inv}_{F'}(\mathrm{Res}\, u_{K/F}) = [F' : F]\,\mathrm{Inv}(u_{K/F}) = [F' : F]\frac{1}{n} = \frac{1}{m} = \mathrm{Inv}_{F'}(u_{K/F'})$$

We have:

$$u_{K/F'} = \mathrm{Res}(u_{K/F}) \tag{9}$$

we know that $G$-module $K^\times$ is a class module with $u_{K/F}$ as its fundamental class. We can now use Tate-Nakayama to get:

**Theorem 2.7.** *For all $n \in \mathbb{Z}$, morphism given by cup product $\alpha \mapsto \alpha \cup u_{K/F}$ is an isomorphism from $\widehat{H}^n(G, \mathbb{Z})$ to $\widehat{H}^{n+2}(G, K^\times)$. Further, we have commutative diagram:*

$$\begin{array}{ccc} \widehat{H}^n(G,\mathbb{Z}) & \xrightarrow{\cup u_{K/F}} & \widehat{H}^{n+2}(G,K^\times) \\ \mathrm{Cor}\updownarrow\mathrm{Res} & & \mathrm{Cor}\updownarrow\mathrm{Res} \\ \widehat{H}^n(H,\mathbb{Z}) & \xrightarrow{\cup u_{K/F'}} & \widehat{H}^{n+2}(H,K^\times) \end{array} \tag{10}$$

**Proposition 2.8.** *Let $K/F$ be finite Galois extension of Local field $F$, with Galois group $G$, then*
*(1) there exists an open subgroup $V$ of $U_K$ such that $\widehat{H}^n(G, V) = 0$, $\forall n \in \mathbb{Z}$.*
*(2) If the extension is unramified, then $\widehat{H}^n(G, U_K) = 0$, $\forall n \in \mathbb{Z}$.*

*Proof.* See [11] Chapter IV. □

Now we do some calculations:

**Proposition 2.9.** *If $F$ is nonarchimedean local field, $K/F$ is unramified Galois extension, $G = G(K/F)$. If $A$ is a finitely generated $\mathbb{Z}$-free module and at the same time a $G$-module. Then the norm morphism induces a surjective morphism:*

$$N_G : \mathrm{Hom}(A, U_K) \to \mathrm{Hom}_G(A, U_K)$$

*Proof.* For $n \geqslant 1$, let $U_K^n = \{x \in U_K \mid x \equiv 1 \mod P_K^n\}$, they are all $G$-invariant. We only need to verify:

$$N_G : \mathrm{Hom}(A, U_K/U_K^1) \to \mathrm{Hom}_G(A, U_K/U_K^1)$$

$$N_G : \mathrm{Hom}(A, U_K^n/U_K^{n+1}) \to \mathrm{Hom}_G(A, U_K^n/U_K^{n+1})$$

are surjective.

Let $k_K = O_K/P_K$ be the residue field of $O_K$. $U_K^n/U_K^{n+1} \cong k_K$ as $G$-module. Then we consider:

$$N_G : \mathrm{Hom}(A, k_K) \to \mathrm{Hom}_G(A, k_K)$$

Assume $k_F = O_F/P_F$, then $k_F$ is isomorphic to $\mathbb{Z}[G]\otimes k_F$ as $G$-module. And $\mathrm{Hom}(A, \mathbb{Z}[G]\otimes k_F) \cong \mathbb{Z}[G] \otimes \mathrm{Hom}(A, k_F)$, so

$$\widehat{H}^0(G, \mathbb{Z}[G] \otimes \mathrm{Hom}(A, k_F)) = 0$$

that is to say, $N_G$ is surjective.

$U_K/U_K^1$ as $G$-module is isomorphic to $k_K^\times$. We consider:

$$N_G : \mathrm{Hom}(A, k_K^\times) \to \mathrm{Hom}_G(A, k_K^\times)$$

we want to show $\widehat{H}^0(G, \mathrm{Hom}(A, k_F^\times)) = 0$. Since $G$ is a finite cyclic group and $\mathrm{Hom}(A, k_K^\times)$ is finite, so all $\widehat{H}^p(G, \mathrm{Hom}(A, k_F^\times))$ have the same order. We shall prove:

$$\widehat{H}^1(G, \mathrm{Hom}(A, k_F^\times)) = 0$$

Let $\overline{k}_K$ be the algebraic closure of $k_K$, $\mathcal{F}$ is the subgroup of $\mathrm{Gal}(\overline{k}_K/k_K)$ generated by the Frobenius automorphism $\sigma_0 : x \mapsto x^{|k_K|}$, then the following sequence is exact:

$$0 \longrightarrow H^1(G, \mathrm{Hom}(A, k_F^\times)) \longrightarrow H^1(\mathcal{F}, \mathrm{Hom}(A, \overline{k}_F^\times))$$

Then we only need to show $H^1(\mathcal{F}, \mathrm{Hom}(A, \overline{k}_F^\times)) = 0$, that is to say, for any 1-cocycle $f$ of $\mathcal{F}$, there is a $\varphi \in \mathrm{Hom}(A, k_F^\times)$ such that $f(\sigma_0) = \sigma_0\varphi - \varphi$. It is done by linear algebra. See [7] p17.

□

## 2.2. **Weil Group and $L$-Group.** First give some definitions:

$$C_K = \begin{cases} \text{idèle class group} & \text{if } K \text{ algebraic number field} \\ K^\times & \text{if } K \text{ Local field} \end{cases}$$

Now we have a special case of Weil group for our use:

**(Weil group, special case)** If $F$ is local or global field, $K/F$ is Galois extension with Galois group $G_{K/F}$. $(G, G_F, C)$ be a class formation from knowledge of class field theory. Then Weil group is defined in 1.3 has its form as an extension of $G_{K/F}$ through $C_K$:

$$0 \longrightarrow C_K \xrightarrow{i} W_{K/F} \xrightarrow{j} G_{K/F} \longrightarrow 0$$

such that its factor set is a fundamental class $u \in H^2(G_{K/F}, C_K)$.

Now we assume that $F$ and $F'$ are local fields or global fields, with $K$ (resp.$K'$) Galois extension of $F$ (resp.$F'$), $\varphi$ is isomorphism from $K$ to $K'$ which maps $F$ to $F'$.

Moreover we add some conditions: if we require $F$ and $F'$ to be simultaneously local fields or global fields, we require $F'$ to be separable over image of $F$; if $F$ is global but $F'$ is local, then require $F'$ to be separable over the closure of image of $F$.

Under these conditions, for $\varphi$ we can associate a homomorphism:

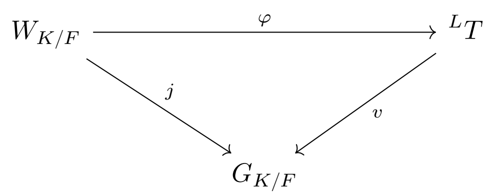
$$\varphi_W : W_{K'/F} \to W_{K/F}$$

Therefore for discrete $G_{K/F}$-module $A$, we can associate a morphism of cohomological groups:

$$\varphi_W^* : H^1_{ct}(W_{K/F}, A) \to H^1_{ct}(W_{K'/F'}, A) \tag{11}$$

## 2.3. **$L$-group of torus.** Assume $F$ is local field or global field, $T$ is an algebraic torus defined over $F$ and splits over Galois extension $K$, $X(T)$ is $G_{K/F}$-module formed by characters of $T$. Let $X_*(T) = \mathrm{Hom}(X(T), \mathbb{Z})$

2.4. **$L$-homomorphisms from Weil Groups to $L$.** We consider continuous homomorphism $\varphi : W_{K/F} \to {}^L T$, such that the following diagram commutes:




For two continuous homomorphism $\varphi$ and $\varphi'$, if exists a $t \in {}^L T^0(\mathbb{C})$ such that $\varphi(w) = t^{-1}\varphi(w)t$, then we say that $\varphi$ and $\varphi'$ are isomorphic. Denote the set of equivalence class of such homomorphism $\Phi(T)$.

If we denote $\varphi(w) = (a(w), j(w))$, where $a(w) \in {}^L T^0$, then $w \mapsto a(w)$ is continuous 1-cocycle from $W_{K/F}$ to ${}^L T^0$. We have

$$t^{-1} \cdot (t' \rtimes \sigma) \cdot t = t^{-1} \cdot t' \cdot {}^{\sigma}t \rtimes \sigma \quad (t,\ t' \in {}^L T^0) \tag{12}$$

Therefore, $\varphi \equiv \varphi'$ if and only if $a$ and $a'$ represent the same cohomological class. We have:

$$\Phi(T) \cong H^1_{ct}(W_{K/F}, {}^L T^0)$$

2.5. **Unramified equivalent class of homomorphisms.** If $F$ is a local field, if element $[\varphi] \in \Phi(T)$ such that $\varphi|_{\text{Inertia Group}}$ is trivial, we call $[\varphi]$ is unramified, we use $\Phi_{\text{unr}}(T)$ to denote all unramified elements of $\Phi(T)$.

If moreover, $K/F$ is assumed to be unramified, then $G_{K/F}$ is generated by Frobenius automorphism $\sigma_0$. Unramified $\varphi$ is determined by $\varphi(1 \times \sigma_0) = t \times \sigma$ completely, where $t \in {}^L T^0$ is determined up to conjugation. Therefore in this case we have:

$$\Phi_{\text{unr}}(T) = ({}^L T^0 \rtimes \sigma)/\operatorname{Int} {}^L T^0 \tag{13}$$

where $\operatorname{Int} {}^L T^0$ represents conjugation group with respect to ${}^L T^0$.

## 3. Representation and Local $L$-function

3.1. **Representation of Torus.** If $F$ is a local field, $T(F)$ is locally compact Abelian group. From Schur's Lemma, we know that: Irreducible representations of $T(F)$ in a Hilbert space are characters, that is to say, continuous homomorphisms $T(F) \to \mathbb{C}^\times$.

For $K$ a global field, from exact sequence:

$$1 \longrightarrow K^\times \longrightarrow J_K \longrightarrow C_K \longrightarrow 1$$

we derive an exact sequence:

$$1 \longrightarrow T(F) \longrightarrow T(\mathbb{A}_F) \longrightarrow \operatorname{Hom}_{G_{K/F}}(X(T), C_K) \longrightarrow H^1(G_{K/F}, T(K))$$

Therefore $C_F(T) = T(\mathbb{A}_F)/T(F)$ can be seen as subgroup of $\operatorname{Hom}_{G_{K/F}}(X(T), C_K)$, to study representations of $T(F)$ ($F$ local field) or representations of $T(\mathbb{A}_F)/T(F)$ ($F$ global field), we need to study the following group:

$$\Pi(T) = \operatorname{Hom}_{ct}(\operatorname{Hom}_{G_{K/F}}(X(T), C_K), \mathbb{C}^\times)$$

*Remark.* We can also consider the character taken values in complex numbers of absolute value 1, see [9] Chapter 1 Section 8.

### 3.2. **Torus Theorem.**

**Theorem 3.1.** *There exists a canonical isomorphism:*

$$\Phi(T) \cong \Pi(T)$$

And its improved version:

**Theorem 3.2.** *(1) If $F$ is a local field, then $H^1_{ct}(W_{K/F}, {}^LT^0)$ is canonically isomorphic to character group of $T(F)$.*

*(2) If $F$ is a global field, then we have a canonical homomorphism from $H^1_{ct}(W_{K/F}, {}^LT^0)$ to character group of $T(\mathbb{A}_F)/T(F)$, with finite kernel, and formed by the following class $\alpha$: when $K'$ is the completion of $K$ with respect to some valuation, we have $\varphi^*_W(\alpha) = 0$, where $F'$ is the algebraic closure of $F$ in $K'$, $\varphi : K/F \to K'/F'$ is an embedding.*

3.3. **Equivalent classes of unramified homomorphisms and characters.** For this subsection, we fix: $T$ is a torus defined over a nonarchimedean local field, and splits over unramified extension $K/F$ with Galois group $G_{K/F}$, let $\sigma_0$ denotes the Frobenius automorphism of $G_{K/F}$.

If a character is trivial over $T(O_F) = \mathrm{Hom}_{G_{K/F}}(X(T), U_K)$, then it is called unramified. The set of unramified characters of $T(F)$ is denoted as $\Phi_{\mathrm{unr}}(T)$.

The exact sequence:

$$0 \longrightarrow U_K \longrightarrow C_K \xrightarrow{v} \mathbb{Z} \longrightarrow 0$$

where $v(a) = 1$ if and only if $a$ generates prime ideal $P_K$. As $G_{K/F}$-module it splits and leads to the following exact sequence:

$$0 \longrightarrow \mathrm{Hom}_{G_{K/F}}(X(T), U_K) \longrightarrow \mathrm{Hom}_{G_{K/F}}(X(T), C_K) \longrightarrow \mathrm{Hom}_{G_{K/F}}(X(T), C_K) \longrightarrow 0$$

We immediately have:

**Lemma 3.3.** *If the character of* $\mathrm{Hom}_{G_{K/F}}(X(T), C_K) = T(F)$ *is trivial on* $\mathrm{Hom}_{G_{K/F}}(X(T), U_K) = T(O_F)$*, then it is character of* $\mathrm{Hom}_{G_{K/F}}(X(T), \mathbb{Z}) = X_*(T)^{G_{K/F}}$*, and is contained in* $\mathrm{Hom}(X(T), \mathbb{Z}) = X_*(T)$.

Using the above notations, we can describe the corollary of Theorem 3.2 (1).

**Corollary 1.** *(1) $\chi \in \Pi(T)$ is unramified if and only if its related element $[f] \in H^1_{ct}(W_{K/F}, {}^LT^0)$ is the lifting of the following:*

$$H^1_{ct}(\mathbb{Z}, {}^LT^0) \to H^1_{ct}(W_{K/F}, {}^LT^0),$$

*this lifting is induced by the following exact sequence:*

$$0 \longrightarrow U_K \longrightarrow W_{K/F} \xrightarrow{\mu} \mathbb{Z} \longrightarrow 0$$

*where $\mu$ satisfies the following conditions: $\mu(w) = 1$ implies $j(w) = \sigma_0$.*

*(2) Besides, if $\chi$ extends trivially to a character of $X_*(T)$ and $\mu(w_0) = 1$, then for $\lambda \in {}^LT^0(\mathbb{C})$ we have*

$$f(w_0)(\lambda) = \chi(\lambda)$$

*(3) Isomorphism $\Phi(T) \cong \Pi(T)$ induces bijection between $\Pi_{\mathrm{unr}}(T)$ and $\Pi_{\mathrm{unr}}(T)$.*

## 4. Proof of Theorem 3.2

To simplify notations, in this section 4, we shall use $C$, $W$, $G$ to denote $C_K$, $W_{K/F}$, $G_{K/F}$. Therefore we have an exact sequence:

$$0 \longrightarrow C \xrightarrow{i} W \xrightarrow{j} G \longrightarrow 0$$

and we can choose right coset representatives of $C$ in $W$ :$\{w_\sigma \mid \sigma \in G\}$, for fixed $\sigma$, $\tau \in G$, $\exists c_{\sigma,\tau} \in C$ such that:

$$w_\sigma w_\tau = c_{\sigma,\tau} w_{\sigma\tau},$$

and the fundamental class $u \in H^2(G, C)$ is 2-cocycle of $c_{\sigma,\tau}$.

### 4.1. **Step 1:**$H_1(C, X_*(T))^G \xrightarrow{\sim} \mathrm{Hom}_G(X(T), C)$.

**Theorem 4.1.** *Prove that there is a $G$-isomorphism:*

$$H_1(C, X_*(T))^G \xrightarrow{\sim} \mathrm{Hom}_G(X(T), C) \tag{14}$$

*Proof.* From cup product:

$$\langle X(T), X_*(T)\rangle \to \mathbb{Z}, \quad \langle \lambda, \widehat{\lambda}\rangle = \widehat{\lambda}(\lambda)$$

we get a bilinear morphism:

$$H^0(C, X(T)) \times H_1(C, X_*((T))) \to H_1(C, \mathbb{Z})$$

it commutes with the action of $G$ on these three groups. Since $H^0(C, X(T))$ and $H_1(C, \mathbb{Z})$ are isomorphic to $X(T)$ and $C$ as $G$-modules, we have isomorphism

$$H_1(C, X_*(T)) \to \mathrm{Hom}(X(T), C) \tag{15}$$

From Proposition 1.3.7, it maps 1-cycle $y$ to the class of the following homomorphisms:

$$\lambda \to \prod_{c \in C} c^{\langle \lambda, y(c)\rangle} \tag{16}$$

Since $X(T)$ is direct sum of $\mathbb{Z}$, this is an isomorphism.

□

### 4.2. **Step 2:** $H_1(W, X_*(T)) \xrightarrow{\sim} H_1(C, X_*(T))^G$.

**Theorem 4.2.** *The transform from $W$ to $C$ leads to an isomorphism:*

$$H_1(W, X_*(T)) \xrightarrow{\sim} H_1(C, X_*(T))^G \tag{17}$$

*Proof.* From definition we know that:

$$H_1(C, X_*(T))^G / N_G(H_1(C, X_*(T))) = \widehat{H}^0(G, H_1(C, X_*(T))).$$

Using isomorphism 15, we have an exact sequence:

$$0 \longrightarrow N_G(H_1(C, X_*(T))) \longrightarrow H_1(C, X_*(T))^G \longrightarrow \widehat{H}^0(G, \mathrm{Hom}(X(T), C)) \longrightarrow 0 \tag{18}$$

From 2.6, we have exact sequence:

$$0 \longrightarrow Z \longrightarrow H_1(C, X_*(T)) \longrightarrow H_1(W, X_*(T)) \longrightarrow H_1(G, X_*(T)) \longrightarrow 0 \tag{19}$$

where $Z = \mathrm{Ker}\,(N_G : H_1(C, X_*(T))) \to H_1(C, X_*(T))$.

We have an obvious isomorphism:

$$X_*(T)\otimes C \xrightarrow{\sim} \mathrm{Hom}(X(T),C) \tag{20}$$

It maps $\widehat{\lambda}\otimes c$ to morphism $\lambda\mapsto c^{\langle\lambda,\widehat{\lambda}\rangle}$, with respect to this pairing, we have cup product:

$$H_1(G,X_*(T))\times \widehat{H}^2(G,C)\to \widehat{H}^0(G,\mathrm{Hom}(X(T),C))$$

According to Tate-Nakayama Theorem, cup product with fundamental class $u\in\widehat{H}^2(G,C)$ gives an isomorphism:

$$E: H_1(G,X_*(T)) \xrightarrow{\sim} \widehat{H}^0(G,\mathrm{Hom}(X(T),C)) \tag{21}$$

According to proposition 2.4, this morphism maps 1-cycle $z$ of $G$ in $X_*(T)$ to class of homomorphism

$$\lambda\mapsto\prod_{\sigma,\tau} c_{\tau,\sigma}^{\langle\lambda,\tau z(\sigma)\rangle} \tag{22}$$

If we combine exact sequences 18, 19 and isomorphism 21, we get a commutative diagram:

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ 0 \to & Z & \to & H_1(C,X_*(T)) & \to & H_1(W,X_*(T)) & \to & H_1(G,X_*(T)) & \to 0 \\ & & & \downarrow N_G & & \downarrow \mathrm{Res} & & \downarrow E & \\ & 0 & \to & N_G(H_1(C,X_*(T))) & \to & H_1(C,X_*(T))^G & \to & \widehat{H}^0(G,\mathrm{Hom}(X(T),C)) & \to 0 \\ & & & \downarrow & & & & \downarrow & \\ & & & 0 & & & & 0 & \end{array} \tag{23}$$

The commutativity of left block is from proposition 2.3.

Fixing a 1-cycle of $W$ in $X_*(T)$, $x: w\mapsto x(w)$, for $\tau\in G$, $s\in W$, exists a unique element $c_{\tau,w}$ and unique $\sigma\in G$ such that $w_\tau w=c_{\tau,w}w_\sigma$. From the proof of proposition 2.3 $\mathrm{Res}(x)$ is the 1-cycle class of the following:

$$y: c\mapsto \sum_{c_{\tau,w}=c} w_\tau x(w)$$

from 16, this cycle's image in $\widehat{H}^0(G,\mathrm{Hom}(X(T),C))$ is the class formed by:

$$\lambda\mapsto\prod_{\tau,w} c_{\tau,w}^{\langle\lambda,w_\tau x(w)\rangle} \tag{24}$$

If $w=cw_\sigma$, $c\in C$, then $c_{\tau,w}=w_\tau c w_\tau^{-1}c_{\tau,\sigma}$, therefore this product equals to

$$\left\{\prod_{\sigma,\tau,c}(w_\tau c w_\tau^{-1})^{\langle\lambda,w_\tau x(cw_\sigma)\rangle}\right\}\left\{\prod_{\sigma,\tau,c} c_{\tau,\sigma}^{\langle\lambda,w_\tau x(cw_\sigma)\rangle}\right\}$$

First product is a norm, this means if we let:

$$z(\sigma)=\sum_c x(cw_\sigma)$$

Then homomorphism 24 have the same cohomological class as:

$$\lambda\mapsto\prod_{\sigma,\tau} c_{\tau,\sigma}^{\langle\lambda,\tau z(\sigma)\rangle} \tag{25}$$

But $z$ is the image of $x$ under the following homomorphism:

$$H_1(W, X_*(T)) \to H_1(G, X_*(T))$$

However from 22, $E(z)$ is the class of 25, so we have proved the commutativity of right square of 23. Therefore by snake lemma, we know 17 is an isomorphism. □

### 4.3. Step 3: $H^1_{ct}(W, {}^LT^0) \xrightarrow{\sim} \mathrm{Hom}(H_1(W, X_*(T)), \mathbb{C}^\times)$.

**Theorem 4.3.** *The pairing associated to valuation map $(t, \lambda) \mapsto \lambda(t)$ ($t \in {}^LT^0$, $\lambda \in X_*(T)$)*

$$H^1_{ct}(W, {}^LT^0) \times H_1(W, X_*(T)) \to \mathbb{C}^\times$$

*leads to an isomorphism:*

$$H^1_{ct}(W, {}^LT^0) \xrightarrow{\sim} \mathrm{Hom}(H_1(W, X_*(T)), \mathbb{C}^\times) \tag{26}$$

*Proof.* We already have $H_1(W, X_*(T))$ isomorphic to $\mathrm{Hom}_G(X(T), C)$, this isomorphism can be used to transform $H_1(W, X_*(T))$ into a topological group.

Because $\mathbb{C}^\times$ is $\mathbb{Z}$-injective, from proposition 2.5, we have isomorphism

$$\Phi : H^1(W, {}^LT^0) \xrightarrow{\sim} \mathrm{Hom}(H_1(W, X_*(T)), \mathbb{C}^\times)$$

To prove 26, we only need to prove $\Phi([f])$ is continuous if and only if $f$ is a continuous cocycle.

Let $U$ denote the set formed by elements of norm 1, then we have exact sequence:

$$1 \longrightarrow U \longrightarrow C \longrightarrow M \longrightarrow 1$$

where $M$ is $\mathbb{Z}$ or $\mathbb{R}$, $G$ acts trivially on it, this sequence splits as an sequence of Abel groups, and the following is exact:

$$0 \longrightarrow \mathrm{Hom}(X(T), U) \xrightarrow{\lambda} \mathrm{Hom}(X(T), C) \xrightarrow{\mu} \mathrm{Hom}(X(T), M) \longrightarrow 0$$

**Proposition 4.4.** *We have an injective morphism:*

$$\psi : (N_G(\mathrm{Hom}(X(T), C)) \cap \mathrm{Hom}(X(T), U))/N_G(\mathrm{Hom}(X(T), U)) \to$$
$$\widehat{H}^{-1}(G, \mathrm{Hom}(X(T), M))/\mu\widehat{H}^{-1}(G, \mathrm{Hom}(X(T), C))$$

Proof of this proposition:

*Proof.* If $z = N_G x \in \mathrm{Hom}(X(T), U)$, $x \in \mathrm{Hom}(X(T), C)$, and $y = \mu(x)$, then $N_G(y) = N_G(\mu(x)) = \mu(N_G x) = 0$. Thus we define the morphism $\psi$ to be the map sending $z$ to the quotient image $\overline{y}$ of $y$ on the right hand side. This is well defined: if $x$ has value in $\overline{\mathrm{Hom}(X(T), U)}$, it is 0. If $x$ and $x'$ satisfy $N_G x = N_G x'$, we have $x - x' = r$, so $r \in \mathrm{Hom}(X(T), U)$, $\overline{\mu(x)} = \overline{\mu(x') + \mu(r)} = \overline{\mu(x')}$.

**Injectivity**: We need to show that if $\psi(z) = 0$ for $z = N_G x$, and $x \in \mathrm{Hom}(X(T), C_K)$, then $\exists x' \in \mathrm{Hom}(X(T), U)$ such that $N_G x = N_G x'$.

If the image is 0, since $y \in I_G \mathrm{Hom}(X(T), M)$, we can choose $x$ such that $y = \sum_\sigma(\sigma^{-1} v_\sigma - v_\sigma)$ for $v_\sigma \in \mathrm{Hom}(X(T), M)$, let $u_\sigma$ be the elements in $\underline{\mathrm{Hom}(X(T), C)}$ such that $\mu(u_\sigma) = v_\sigma$, then $x' = x - \sum_\sigma(\sigma^{-1} u_\sigma - u_\sigma) \in \mathrm{Hom}(X(T), U)$ and $N_G x = N_G x'$, $\overline{\mu(x')} = \overline{\mu(x)} = 0$. □

Now we can show $N_G(\mathrm{Hom}(X(T), C))$ is closed in $\mathrm{Hom}_G(X(T), C)$.

**Case 1**:

Since we have $\mathrm{Hom}(X(T), U) \cong T(O_K) \cong (U_K)^d$ where $d$ is rank of lattice $X(T)$, it is compact. Note $N_G$ is a continuous map, so $N_G(\mathrm{Hom}(X(T), U))$ is compact subgroup of $\mathrm{Hom}(X(T), U)$, thus closed in $\mathrm{Hom}(X(T), U)$, hence in $N_G(\mathrm{Hom}(X(T), C)) \cap \mathrm{Hom}(X(T), U)$. And since the above

Proposition gives injectivity of $\psi$, we know $N_G(\mathrm{Hom}(X(T),U))$ is of finite index in $N_G(\mathrm{Hom}(X(T),C))\cap \mathrm{Hom}(X(T),U)$, so $N_G(\mathrm{Hom}(X(T),C))\cap\mathrm{Hom}(X(T),U)$ is closed in $\mathrm{Hom}(X(T),U)$

Except for $K$ archimedean or global, we have $\mathrm{Hom}_G(X(T),U)$ is open in $\mathrm{Hom}_G(X(T),C)$, and

$$N_G(\mathrm{Hom}(X(T),C))\cap\mathrm{Hom}_G(X(T),C) = N_G(\mathrm{Hom}(X(T)),C)\cap\mathrm{Hom}(X(T),U)$$

is closed. From knowledge of topological groups, we know $N_G(\mathrm{Hom}(X(T),C))$ is closed in $\mathrm{Hom}_G(X(T),C)$.

It is also open because $M$ discrete.

**Case 2**:

In the archimedean or global field case,

$$1 \longrightarrow U \longrightarrow C \longrightarrow M = \mathbb{R}^{>0} \longrightarrow 1$$

splits as a $G$-module, we have the following split exact sequence:

$$0 \longrightarrow \mathrm{Hom}(X(T),U) \xrightarrow{\lambda} \mathrm{Hom}(X(T),C) \xrightarrow{\mu} \mathrm{Hom}(X(T),M) \longrightarrow 0$$

So we have

$$\mathrm{Hom}(X(T),C) \cong \mathrm{Hom}(X(T),U)\times\mathrm{Hom}(X(T),M) \tag{27}$$

and

$$N_G(\mathrm{Hom}(X(T),C)) \cong N_G(\mathrm{Hom}(X(T),U))\times N_G(\mathrm{Hom}(X(T),M))$$

We also have:

$$\mathrm{Hom}_G(X(T),C)\cong\mathrm{Hom}_G(X(T),U)\times\mathrm{Hom}_G(X(T),M)$$

Since $M=\mathbb{R}^{>0}$ is divisible, we have $\widehat{H}^0(G,\mathrm{Hom}(X(T),M))=0$, which means:

$$N_G(\mathrm{Hom}(X(T),M)) = \mathrm{Hom}_G(X(T),M)$$

Combined with the fact that $N_G(\mathrm{Hom}(X(T),U))$ is closed in $\mathrm{Hom}_G(X(T),U)$, we see $N_G(\mathrm{Hom}(X(T),C))$ is closed in $\mathrm{Hom}_G(X(T),C)$.

It is also open in it because $N_G(\mathrm{Hom}(X(T),U))$ is of finite index in $\mathrm{Hom}_G(X(T),U)$. Now we have: for $\varphi\in\mathrm{Hom}_G(X(T),C)$, it is continuous if and only if $\varphi\circ N_G$ is continuous.

We have the following lemma which can be proved easily:

**Lemma 4.5.** *A 1-cocycle $x$ of $H^1(W,{}^LT^0)$ is continuous if and only if its restriction to $H^1(C,{}^LT^0)$ is continuous.*

The following diagram is commutative:

$$\begin{array}{ccc} H^1(W,{}^LT^0) & \xrightarrow{\sim} & \mathrm{Hom}(H_1(W,X_*(T)),\mathbb{C}^\times) \\ \downarrow{\scriptstyle \mathrm{Res}} & & \downarrow{\scriptstyle \widehat{\mathrm{Cor}}} \\ H^1(C,{}^LT^0) & \xrightarrow{\sim} & \mathrm{Hom}(H_1(C,X_*(T)),\mathbb{C}^\times) \end{array}$$

where $\widehat{\mathrm{Cor}}$ is induce by $\mathrm{Cor}: H_1(C,X_*(T))\to H_1(W,X_*(T))$. $[f]\in Z^1(C,{}^LT^0)$ under the bottom morphism $E$ is the map sending $\widehat{\lambda}\otimes a\in\mathrm{Hom}(X_*(T)\otimes C,\mathbb{C}^\times)\cong\mathrm{Hom}(H_1(C,X_*(T)),\mathbb{C}^\times)$ to $\langle\widehat{\lambda},f(a)\rangle$, this is continuous.

4.3.1. *Proof of Theorem 3.2 (2).* This part, I mainly follow [9] Chapter 1 Section 8 and [8].

Here are some preparations:

Let $F$ be a global or local field, and let $K$ be a finite Galois extension of $F$. Let $M$ be a finitely generated torsion free $G_{K/F}$-module, then we define:

$$\begin{aligned} M' &:= \mathrm{Hom}_{ct}(M, \mathbb{C}^\times) \\ M^\dagger &:= \mathrm{Hom}(M, \mathbb{C}^\times) \end{aligned} \tag{28}$$

They are again $G_{K/F}$-modules, we regard these groups as $W_{K/F}$-modules. If we write $W_{K/F} = \bigsqcup w_g C_K$ as union of disjoint left cosets. As constructed in [8] section 3, we define:

$$\mathrm{Cor} : H^1(C_K, M^\dagger) \to H^1(W_{K/F}, M^\dagger)$$

as the map sending $\alpha : C_K \to M^\dagger$ to map $\mathrm{Cor}(\alpha) : W_{K/F} \to M^\dagger$ such that

$$(\mathrm{Cor}(\alpha))(w) = \sum_{g \in G} w_g \alpha(w_g^{-1} w w_{g'}), \quad \text{where } w w_{g'} \equiv w_g \bmod C_K$$

From definition of Weil groups 1.3, let $(G, G_F, C)$ be a class formation, if we let $G_K$ denote an open normal subgroup of finite index of $G$, $C_F = C^G$. then we have:

$$0 \longrightarrow C_K \xrightarrow{\ i\ } W_{K/F} \xrightarrow{\ j\ } G_{K/F} \longrightarrow 0$$

And it corresponding to the canonical class $u \in H^2(G_{K/F}, C_K)$.

For any $W_{K/F}$-module $M$, the Hochschild-Serre spectral sequence gives an exact sequence:

$$0 \longrightarrow H^1(G_{K/F}, M^\dagger) \xrightarrow{\mathrm{Inf}} H^1(W_{K/F}, M^\dagger) \xrightarrow{\mathrm{Res}} H^1(C_K, M^\dagger)^{G_{K/F}} \xrightarrow{\ \tau\ } H^2(G_{K/F}, M^\dagger)$$

we can make the last morphism $\tau$ (called the transgression) explicitly in our case:

**Lemma 4.6.** *If $C_K$ acts trivially on $M$, then the transgression*

$$\tau : H^1(C_K, M^\dagger)^{G_{K/F}} \to H^2(G_{K/F}, M^\dagger)$$

*is the negative of the map $- \cup u$ induced by the pairing*

$$\mathrm{Hom}(C_K, M) \times C_K \to M$$

*Proof.* Write $W_{K/F} = \bigsqcup_g C_K w_g$, and let $w_g w_{g'} = c_{g,g'} w_{gg'}$. Then $(c_{g,g'})$ is a 2-cocycle representing $u$. Let $\alpha \in \mathrm{Hom}_{G_{K/F}}(C_K, M)$ and define $\beta(c w_g) = \alpha(c)$, $c \in C_K$. Then

$$\begin{aligned} d\beta(g, g') &:= d\beta(w_g, w_{g'}) \\ &= g\beta(w_{g'}) - \beta(w_g w_{g'}) + \beta(w_g) \\ &= -\alpha(c_{g,g'}) \end{aligned} \tag{29}$$

□

which equals $-(\alpha \cup u)(g, g')$.

**Lemma 4.7.** *The corestriction map* $\mathrm{Cor} : H^1(C_K, M^\dagger) \to H^1(W_{K/F}, M^\dagger)$ *factors through* $H^1(C_K, M^\dagger)_{G_{K/F}}$.

*Proof.*

$$\mathrm{Cor}(h\alpha)(w) = \sum_g w_g w_h \alpha(w_h^{-1} w_g^{-1} w w_{g'} w_h)$$

where $g'$ is such that $ww_{g'} \equiv w_g \mod C_K$. $w(w_{g'}w_h) \equiv (w_g w_h) \bmod C_K$. Therefore the class of $\mathrm{Cor}(h\alpha)$ is the same as that of $\mathrm{Cor}(\alpha)$, so $\mathrm{Cor}((h-1)\alpha) = 0$ in $H^1(W_{K/F}, M^\dagger)$. □

**Lemma 4.8.** *The composite*

$$H^1(C_K, M^\dagger) \xrightarrow{\mathrm{Cor}} H^1(W_{K/F}, M^\dagger) \xrightarrow{\mathrm{Res}} H^1(C_K, M^\dagger)$$

*is equal to* $N_G$.

*Proof.* When $w \in C_K$, for $\alpha \in Z^1(C_K, M^\dagger)$ and $w \in W_{K/F}$,

$$\mathrm{Cor}(\alpha)(w) = \sum_g g\alpha(g^{-1}wg) = (N_G\alpha)(w).$$

□

**Theorem 4.9.** *For any finitely generated torsion free $G_{K/F}$-module $M$, the corestriction map defines an isomorphism:*

$$\Phi : \mathrm{Hom}_{ct}(C_K, \mathrm{Hom}_{ct}(M, \mathbb{C}^\times))_{G_{K/F}} \xrightarrow{\approx} H^1_{ct}(W_{K/F}, \mathrm{Hom}_{ct}(M, \mathbb{C}^\times))$$

*Proof.* Write $G$ in short for $G_{K/F}$. First proof that the corestriction defines an isomorphism

$$\mathrm{Hom}(C_K, M')_{G_{K/F}} \to H^1(W_{K/F}, M^\dagger)$$

and then shows that is makes continuous homomorphisms correspond to continuous cocycles.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{H}^{-1}(G, \mathrm{Hom}(C_K, M^\dagger)) & \longrightarrow & \mathrm{Hom}(C_K, M^\dagger)_G & \xrightarrow{N_G} & \mathrm{Hom}(C_K, M^\dagger)^G & \longrightarrow & \widehat{H}^0(G, \mathrm{Hom}(C_K, M^\dagger)) \\ & & \downarrow \approx & & \downarrow \mathrm{Cor} & & \mathrm{Id} \downarrow & & \downarrow \approx \\ 0 & \longrightarrow & H^1(G, M^\dagger) & \xrightarrow{\mathrm{Inf}} & H^1(W_{K/F}, M^\dagger) & \xrightarrow{\mathrm{Res}} & H^1(C_K, M^\dagger)^G & \longrightarrow & H^2(G, M^\dagger) \end{array} \tag{30}$$

The horizontal line is the definition sequence of Tate cohomology groups, the bottom line is Hochschild-Serre spectral sequence, the two vertical isomorphisms are consequences of Tate-Nakayama, the third square commutes because of lemma 4.6. The second square commutes because of lemma 4.8. The first square commutes by explicitly calculating each maps, see [9] Lemma 8.7. By five lemma, Cor in 30 is an isomorphism.

□

Next we show it makes continuous homomorphisms correspond to continuous:
The following is from [8] section 5 and [9] lemma 8.10.

**Proposition 4.10.** *If $D$ is an (real) abelian connected Lie group, equipped with an action of $G = G_{K/F}$ (analytic) then the natural homomorphism:*

$$\widehat{H}^p(G, \mathrm{Hom}_{ct}(C_K, D)) \to \widehat{H}^p(G, \mathrm{Hom}(C_k, D))$$

*is an isomorphism for all $p \in \mathbb{Z}$.*

*Proof.* (a) $K$ and $F$ are local archimedean. The only nontrivial case is $F = \mathbb{R}$ and $K = \mathbb{C}$, here $C_K = \mathbb{C}^\times$, the exact sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R}^2 \longrightarrow \mathbb{C}^\times \longrightarrow 0$$

gives exact sequence:

$$0 \longrightarrow \mathrm{Hom}_{ct}(\mathbb{C}^\times, D) \longrightarrow \mathrm{Hom}_{ct}(\mathbb{R}^2, D) \longrightarrow \mathrm{Hom}_{ct}(\mathbb{Z}, D) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Hom}(\mathbb{C}^\times, D) \longrightarrow \mathrm{Hom}(\mathbb{R}^2, D) \longrightarrow \mathrm{Hom}(\mathbb{Z}, D) \longrightarrow 0$$

because $D$ is divisible hence $\mathbb{Z}$-injective and it is an abelian connected Lie group. From $\mathrm{Hom}_{ct}(\mathbb{C}, D)$ and $\mathrm{Hom}(\mathbb{C}, D)$ cohomologically trivial, we know that we can replace $C_F$ with $\mathbb{Z}$ which is discrete and it is obvious.

(b) $F$ and $K$ nonarchimedean local fields. If $U_K$ the group of units of $K^\times$, we have $K^\times/U_K \cong q^{\mathbb{Z}}$; and if $U_K^n$ is the subgroup of units congruent to 1 module $n$-th power of maximal ideal, we know from [10] Chapter XII, section 3 that $U_K^1$ and $U_K^1/U_K^n$ are cohomologically trivial for all $n$ if $K/F$ is unramified. We know that if $A$ is cohomologically trivial and $D$ is divisible, then $\mathrm{Hom}(A, D)$ is also cohomologically trivial. So $\mathrm{Hom}(U_K^1, D)$ and $\mathrm{Hom}(U_K^1/U_K^n, D)$ are cohomologically trivial. Because

$$\mathrm{Hom}_{ct}(U_K^1, D) = \varinjlim \mathrm{Hom}(U_K^1/U_K^n, D)$$

we know $\mathrm{Hom}_{ct}(U_K^1, D)$ is cohomologically trivial. So again we can replace $C_K$ by $K^\times/U_K^1$ which is discrete. For the general case, replace $U_K^n$ by $V_K^n$, where $V_K$ is as in proposition 2.8, the proof is similar.

(c) $F$ global. Here $C_K$ is the idèle class group. Define $V \subset C_F$ to be $\prod V_v$ where $V_v = \widehat{\mathcal{O}}_v^\times$ for $v$ nonarchimedean prime that is unramified in $K$, and $V_v$ is a subgroup as in above case for the rest primes. It is therefore enough to prove the lemma for $C_F/V$. In the function field case, this is discrete and in the number field case this is an extension of a finite group by $\mathbb{R}^\times$. In the first case it is done, in the second case by exponential shows that $\mathbb{R}^\times$ is the quotient of a uniquely divisible group by a discrete group.

□

Now we have:

**Corollary 2.** *The map:*

$$\mathrm{Cor} : \mathrm{Hom}_{ct}(C_K, D)^G \to H^1_{ct}(W_{K/F}, D)$$

*is bijective.*

**Proposition 4.11.** *If $\varphi \in Z_c^1(W_{K/F}, D)$ we say $\varphi$ if not ramified is its restriction to $U_K$ is trivial, and we note $H^1_{unr}(W_{K/F}, D)$, which by the proof of proposition 4.10, is isomorphic to* $\mathrm{Hom}(\mathbb{Z}, D)^G$.

And we also derive a lemma:

**Lemma 4.12.** *If $D$ is a compact group, then $I_G \mathrm{Hom}_{ct}(C_K, D)$ is closed in $\mathrm{Hom}_{ct}(C_K, D)$, equipped with compact convergence topology.*

*Proof.* In proposition 4.10, applied to $p = -1$, we have :

$$0 = \mathrm{Ker}\left(\widehat{H}^{-1}(G, \mathrm{Hom}_c(C_K, D)) \to \widehat{H}^{-1}(G, \mathrm{Hom}(C_K, D))\right) =$$

$$\left(I_G \mathrm{Hom}(C_K, D) \bigcap \mathrm{Hom}_{ct}(C_K, D)\right) / I_G \mathrm{Hom}_{ct}(C_K, D)$$

□

What we concern is $D = M' = \mathrm{Hom}(X_*(T), \mathbb{C}^\times)$, following [8] section 6, we separate it into two cases: $\mathbb{R}$ and $\mathbb{R}/\mathbb{Z}$ by the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R}^2 \longrightarrow \mathbb{C}^\times \longrightarrow 0$$

then conclude for $\mathbb{C}^\times$.

**Proposition 4.13.** *We have an isomorphism:*

$$\Gamma : \mathrm{Hom}_{ct}(C_K \otimes M, \mathbb{C}^\times)_G \to \mathrm{Hom}_{ct}\left((C_K \otimes M)^G, \mathbb{C}^\times\right)$$

*Proof.* Now suppose $D = \mathrm{Hom}(M, S)$ where $M = X_*(T)$ is a $Z[G]$-module which as $\mathbb{Z}$-module is free and of finite type, and $S$ is a real connected abelian Lie group where $G$ acts trivially on it. Under this hypothesis, $D$ is a connected abelian Lie group.

We have a natural isomorphism

$$\mathrm{Hom}_{ct}(C_K \otimes X_*(T), S) \to \mathrm{Hom}_{ct}(C_K, \mathrm{Hom}(X_*(T), S))$$

where $C_K \otimes X_*(T) \cong C_K^n$ ($n$ is rank of $X(T)$), equipped with product topology.

Now first suppose $S = \mathbb{R}/\mathbb{Z}$, we see $\mathrm{Hom}_{ct}(C_K, D)$ is just the Pontryagin dual of $C_K \otimes X_*(T)$, then we claim the orthogonal(in sense of topological groups) of the subgroup $(C_K \otimes X_*(T))^G$ in $C_K \otimes X_*(T)$ is the closed subgroup

$$I_G \,\mathrm{Hom}_{ct}(C_K \otimes X_*(T), S)$$

Now prove this claim by showing: for any $\sigma \in G$ and any $f \in \mathrm{Hom}_{ct}(C_K \otimes X_*(T), S)$, $(\sigma f - f)(a) = f(\sigma^{-1}(a)) - f(a) = 0$ if $a = \sigma(a)$ ($a \in C_K \otimes X_*(T)$). So $I_G \,\mathrm{Hom}_{ct}(C_K \otimes X_*(T), S)$ is contained in the kernel of the following restriction:

$$\mathrm{Hom}_{ct}(C_K \otimes X_*(T), S) \to \mathrm{Hom}_{ct}((C_K \otimes X_*(T))^G, S)$$

Conversely, if $a \notin (C_K \otimes X_*(T))^G$, there is a $\sigma \in G$ such that $a \neq \sigma(a)$, since Pontryagin dual separates points, there is a $f$ such that:

$$f(\sigma^{-1}(a) - a) \neq 0$$

so $(\sigma - 1)f(a) \neq 0$, the orthogonal of $I_G \,\mathrm{Hom}_{ct}(C_K \otimes X_*(T), S)$ is included in the orthogonal of the kernel. And we conclude by lemma 4.12.

When $T = \mathbb{R}$, notice that homomorphism to $\mathbb{R}$ is trivial on compact subgroups, we have:

$$\mathrm{Hom}_{ct}(C_K \otimes X_*(T), \mathbb{R})^G = \mathrm{Hom}(X_*(T), \mathbb{R})^G = \mathrm{Hom}(X_*(T)^G, \mathbb{R}) = \mathrm{Hom}_{ct}((C_K \otimes X_*(T))^G, \mathbb{R})$$

□

We have another version of this result, in [9] corollary 8.11. We don't prove it since it is not needed in our use.

**Theorem 4.14.** *Let $F$ be a global or local field, and let $M$ be a finitely generated torsion free $G_F$-module, there is a canonical isomorphism:*

$$\mathrm{Hom}_{ct}((C \otimes M)^{G_F}, \mathbb{C}^\times) \xrightarrow{\approx} H^1_{ct}(W_F, \mathrm{Hom}_{ct}(M, \mathbb{C}^\times))$$

*where $C = \bigcup C_F$.*

Now Let's prove (2) of theorem 3.2.

Recall that, in 3.1, we have:

$$1 \longrightarrow T(F) \longrightarrow T(\mathbb{A}_F) \longrightarrow \mathrm{Hom}_{G_{K/F}}(X(T), C_K) \longrightarrow H^1(G_{K/F}, T(K))$$

(Note that $(X_*(T)\otimes C_K)^{G_K} \cong \mathrm{Hom}_G(X(T), C_K)$.) We have a surjection with finite kernel:

$$\Psi : \mathrm{Hom}_{ct}((X_*(T)\otimes C_K)^G, \mathbb{C}^\times) \to \mathrm{Hom}_{ct}(T(\mathbb{A}_K)/T(K)), \mathbb{C}^\times)$$

Then precompose it first with isomorphism $\Gamma$ in proposition 4.13, then precompose it with isomorphism $\Phi^{-1}$ gotten from Theorem 4.9 taking $M = X_*(T)$:

We get a surjective map $\Psi\circ\Gamma\circ\Phi^{-1}$:

$$H^1_{ct}(W_{K/F}, {}^LT^0) \twoheadrightarrow \mathrm{Hom}_{ct}(T(\mathbb{A}_K)/T(K)), \mathbb{C}^\times)$$

Then the following diagram commutes:

$$\begin{array}{ccc} H^1_{ct}(W_{K/F}, {}^LT^0) & \longrightarrow & \mathrm{Hom}_{ct}(T(\mathbb{A}_K)/T(K), \mathbb{C}^\times) \\ \downarrow & & \hookdownarrow \\ \prod H^1_{ct}(W_{K'/F'}, {}^LT^0) & \xrightarrow{\approx} & \prod \mathrm{Hom}_{ct}(T(K'), \mathbb{C}^\times) \end{array}$$

we know that right vertical and lower horizontal map are injective, so we have the kernels of the remaining maps are the same. The theorem is proved.

□

4.3.2. *Proof of Corollary 1* (1), (2). Now we can prove Corollary 1 (1), (2):

Note that $T$ is torus defined over a nonarchimedean local field $F$, splits over unramified extension $K/F$

*Proof.* There is an exact sequence:

$$0 \longrightarrow U \longrightarrow W \xrightarrow{\mu} \mathbb{Z} \longrightarrow 0$$

such that $\mu(w) = 1$ if and only if the transfer of $w \in C$ generates the prime ideal $P_K$. Since the norm morphism:

$$N_G : \mathrm{Hom}(X(T), U_K) \to \mathrm{Hom}_G(X(T), U_K)$$

is surjection by proposition 2.9, we see that under the isomorphism:

$$H_1(W, X_*(T)) \xrightarrow{\sim} \mathrm{Hom}_G(X(T), C)$$

$H_1(U_K, X_*(T))$ corresponds to $\mathrm{Hom}_G(X(T), U_K)$, so character associated with $H^1_{ct}(W, {}^LT^0)$ is unramified if and only if it is the lifting image of the following:

$$H^1_{ct}(\mathbb{Z}, {}^LT^0) \to H^1_{ct}(W, {}^LT^0) \tag{31}$$

where the action of $\mathbb{Z}$ on ${}^LT^0$ is determined by action of $W$. This proves (1).

If $\widehat{\lambda}$ is an invariant element of $X_*(T)$, $w \in W$, then exists $x$, 1-cocycle of $W$ in $X_*(T)$ such that $x(w) = \widehat{\lambda}$, and when $u \neq w$ we have $x(u) = 0$. The class of $x$ in $\mathrm{Hom}(X(T), C)$ is the morphism:

$$\lambda \mapsto \prod_\tau c_{\tau,w}^{\langle\lambda,\widehat{\lambda}\rangle} \tag{32}$$

Note that $\prod_\tau c_{\tau,w}$ is just the transfer(also called Verlagerung) of $w$. Recall the exact sequence:

$$0 \longrightarrow U \longrightarrow C \xrightarrow{\nu} \mathbb{Z} \longrightarrow 0$$

We apply $\nu$ to above and get:

$$\lambda \mapsto \langle \lambda, \mu(w)\widehat{\lambda} \rangle \tag{33}$$

which is in $\mathrm{Hom}(X(T), \mathbb{Z})$. If $\chi$ is unramified, then is a 1-cocycle $f$ of $W_{K/F}$ with values in ${}^LT^0$ whose lifting is the image of map 31, so for the corresponding homomorphism 33 is given by $\widehat{\lambda} \in X_*(T)^G$ and one $w \in W$ such that $\mu(w) = 1$. And:

$$f(w)(\widehat{\lambda}) = \chi(\widehat{\lambda})$$

□

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