SOME SPECIAL CASES OF KOTTWITZ CONJECTURE

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2. Notations and conventions

First, let us fix some notations through out this article:
k: an algebraically closed field of characteristic p.
K_0 : the fraction field of the Witt ring $W(k)$.
$\overline{K_0}$: an algebraic closure of K_0.
$E \supset F \supset \mathbb{Q}_p$: a sequence of extensions with each one finite over the later one.
$\mathcal{O}_*$: the ring of integers of $*$, ϖ_*: an uniformizer of $\mathcal{O}_*$, $\overline{*}$ (*resp.* $*^{\mathrm{sep}}$, *resp.* $*^{un}$) the algebraic closure (*resp.* separable closure, *resp.* maximal unramified extension) of $*$ taken in $\overline{K_0}$.
$\overline{\mathbb{Q}}_\ell$: a fixed algebraic closure of $\mathbb{Q}_\ell$ for some ℓ prime, $\ell \neq p$.
v_*: normalized valuation of $*$, *i.e.* $v_*(\varpi_*) = 1$.
$\mathbb{F} := \mathcal{O}_F/\varpi_F$ the residue field, and $q = \mathrm{Card}(\mathbb{F})$ the cardinal of $\mathbb{F}$.
$\mathbb{F}_p$: the residue field of $\mathbb{Q}_p$.
$\widehat{\overline{*}}$: the completion of an algebraic closure of $*$, *e.g.* $\mathbb{C}_p := \widehat{\overline{\mathbb{Q}}}_p$.
$\breve{E}$: the completion of the maximal unramified extension of E taken in $\widehat{\overline{K_0}}$, $L := \breve{F}$.
$\Gamma := \mathrm{Gal}(\overline{F}/F)$ the absolute Galois group of F where $\overline{F}$ is the algebraic closure of F taken in $\overline{K_0}$.
σ : the arithmetic Frobenius automorphism of L/F, $I := \mathrm{Gal}(\widehat{\overline{F}}/L) \cong \mathrm{Gal}(\overline{F}/F^{un})$, $I_E := \mathrm{Gal}(\widehat{\overline{E}}/\breve{E})$.
$W_{\overline{F}/F}$ (*resp.* $W'_{\overline{F}/F}$) : the Weil group (*resp.* Weil-Deligne group) of $\overline{F}$ over F (see [Tate]), W_F^{ab}: the abelianized Weil group.
M^Γ: the subgroup of invariants for a Γ-module M, M_Γ : the group of coinvariants of M, M_{tors}: the torsion subgroup of M.

For any reductive group G over F, let G^0 denote its connected component containing identity and $\pi_0(G)$ denote the set of all connected components of G. We assume G is connected if without any other specification. Let G_{ss} denote the derived group of G (it is semisimple) and let G_{sc} denote the universal covering group of G (it is simply connected). $Z(G)$ or abbreviated as Z when no confusion will be caused, is the center of G and set $G_{\mathrm{ad}} := G/Z$. For any field K' containing F, $G_{K'}$ denotes the base change; for any finite extension $E \supset F$, and G a reductive group defined over E, the restriction of scalar $\mathrm{Res}_{E/F}(G)$ is as define in [Mil1] §2.i.. For any G of multiplicative type, *ibid.* §12.f., e.g. a torus, $X^*(G) := \mathrm{Hom}(G, \mathbb{G}_m)$ is called the character group and $X_*(G) := \mathrm{Hom}(\mathbb{G}_m, G)$ is called the cocharacter group (Hom here denotes the morphisms of algebraic groups).

Let us fix more notations when we consider quasi-split groups as this assumption usually makes many definitions explicit and is satisfied in many situations: Let B be a Borel subgroup of G and T a maximal torus contained in B. Let Φ (*resp.* Φ^+, *resp.* Δ^*) be the associated (resp.positive, resp.simple) root system with respect to fixed triple (G, B, T). Let Φ^* denote the coroot system, and other notations with 'co-' follow similarly. Denote by $\Psi_0(G) := (X^*, \Delta^*, X_*, \Delta_*)$ the *based root datum*, [Mil1] Appendix C Definition C.28. Denote by $W := N(T)(L)/Z(T)(L)$ the Weyl group of G where $N(T)(L)$ (*resp.* $Z(T)(L)$) is the L-points of the normalizer (*resp.* centralizer) of T in G. Denote the inner conjugate action by $\mathrm{Int}(g)(\cdot) = g(\cdot)g^{-1}$.

Let G be a connected reductive group defined over F, then Γ acts on $\Psi_0(G_{\overline{F}})$, a *splitting* of a connected reductive group G is a triple $(T, B, \{X_\alpha\}_{\alpha\in\Delta*})$, where T is a maximal torus of G, B is a Borel subgroup of G that contains T, and X_α is a nonzero element of the root space $\mathrm{Lie}(G)_\alpha$, where $\mathrm{Lie}(G)$ is the Lie algebra associated to G, defined in [Mil1] §10.b..

Let (π, V) be any representation of G, we denote the *contragredient representation* of (π, V) by $(\pi^\vee, V^\vee)$ (*cf.* [Mil1] p471). We call a representation (π, V) of G a F-rational representation if π is a rational homomorphism of F-algebraic groups. For any irreducible representation ρ of G, we denote the ρ-isotypical component of V by $V[\rho]$.

3. Introduction

Already by 1930 a great deal was known about (local) class field theory. By work of Kronecker, Weber, Hilbert, Takagi, Artin, Hasse, and others, one could classify the abelian extensions of a local field F, in terms of data which are intrinsic to F. Namely, there is a reciprocity map (or called *Artin map*)

$$\mathrm{Art}_F : F^\times \xrightarrow{\sim} \mathrm{Gal}(F^{ab}/F)$$

where F^{ab} means the maximal abelian extension of F. This map is continuous and has dense image, which is the abelianized Weil group W_F^{ab}.

We can formulate it in a local Langlands correspondence way (special attention should be paid to topology):

Denote by $\mathcal{A}_1(F)$ the set of isomorphism classes of 1-dimensional continuous irreducible complex representations (π, V) of $\mathrm{GL}_1(F)$, that is to say $\mathrm{GL}_1(F) = F^\times \to \mathrm{Aut}_{\mathbb{C}}(\{v\}) = \mathrm{GL}_1(\mathbb{C})$. On the other hand, we denote by $\mathcal{G}_1(F)$ the set of 1-dimensional representations of W_F (continuous homomorphisms) $W_F \to \mathrm{GL}_1(\mathbb{C})$. Then $W_F \to \mathrm{GL}_1(\mathbb{C})$ passes to quotient: $W_F \to W_F^{ab} \to \mathrm{GL}_1(\mathbb{C})$.

Therefore the existence of a reciprocity map is equivalent to the existence of a bijection:

$$\mathcal{A}_1(F) \to \mathcal{G}_1(F)$$

Similar but more refined results called Local Langlands correspondence are proved (for $\mathrm{GL}_n(F)$) or conjectured for all reductive groups.

Now come back to the Artin map, it is natural to ask how to describe Art_F explicitly. It is perfectly solved by Lubin and Tate. Let $\widehat{F}^\times$ be the profinite completion of F, and write $\widehat{F}^\times = U_0 \times \varpi_F^{\widehat{\mathbb{Z}}}$ where $U_0 := \mathcal{O}_F^\times$ is the group of units of F. The fixed field of the image $\mathrm{Art}(U_0)$ is the maximal unramified extension F^{un} with $\mathrm{Gal}(F^{un}/F) \cong \widehat{\mathbb{Z}}$. The fixed field of the image $\mathrm{Art}(\varpi^{\widehat{\mathbb{Z}}})$ is an infinite totally ramified extension of F denoted by F_ϖ with $\mathrm{Gal}(F_\varpi/F) \cong U_0$. For example, $F = \mathbb{Q}_p$, then $\varpi_{\mathbb{Q}_p} = p$, $\mathbb{Q}_p^{ab}$ by Kronecker-Weber theorem equals to $\mathbb{Q}_p^{\mathrm{cycl}}$ the maximal cyclotomic extension of $\mathbb{Q}_p$, *i.e.* the extension by adding all roots of unit.

It is easy to construct $\mathbb{Q}_p^{un}$. Let μ_s be the set of s-th root of 1 in $\overline{\mathbb{Q}}_p$ for some $s \in \mathbb{N}$ such that $(s,p) = 1$. The discriminant of $X^s - 1$ is a unit in $\mathbb{Z}_p$. The field $\mathbb{Q}_p[\mu_s]$ is unramified over $\mathbb{Q}_p$. Moreover, the residue field of $\mathbb{Q}_p[\mu_s]$ is the splitting field of $X^s - 1$ over $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$, so it has p^f elements with f being the smallest positive integer such that $s | p^f - 1$. Therefore $\bigcup_{p \nmid s} \mathbb{Q}_p[\mu_s]$ is an unramified extension with residue field $\overline{\mathbb{F}_p}$.

We can also construct the totally ramified extension. We have $(\mathbb{Q}_p)_p = \bigcup_{n \geqslant 0} \mathbb{Q}_p[\mu_{p^n}(\overline{\mathbb{Z}_p})]$ where μ_{p^n} are the p-torsion parts of $\mathbb{G}_m$ (which is closely related to a function $[p]_f(T) = f(T) = x^p - 1$) and $\overline{\mathbb{Z}_p}$ is closure of $\mathbb{Z}_p$ taken in a fixed algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. We see $\{\mathbb{Q}_p[\mu_{p^n}]\}_{n \geqslant 0}$ as a tower formed by points, that is to say, 0-dimensional varieties. The action

$$([m], \zeta) \mapsto \zeta^m : \mathbb{Z}_p/p^n\mathbb{Z}_p \times \mu_{p^n} \to \mu_{p^n}$$

makes μ_{p^n} a free $\mathbb{Z}_p/p^n\mathbb{Z}_p$-module of rank 1, thus we can regard μ_{p^n} as a $\mathbb{Z}_p$-module, isomorphic to $\mathbb{Z}_p/p^n\mathbb{Z}_p$. There is an isomorphism

$$(\mathbb{Z}_p/p^n\mathbb{Z}_p)^\times \ \to \mathrm{Gal}(\mathbb{Q}_p[\mu_{p^n}]/\mathbb{Q}_p)$$

when passing to limit, we have an isomorphism

$$\mathbb{Z}_p^\times \to \mathrm{Gal}((\mathbb{Q}_p)_p/\mathbb{Q}_p)$$

Move to a different field: When F is a finite extension of $\mathbb{Q}_p$, the above process need to be changed. Those roots of unity $\mu_{p^n}(\overline{\mathbb{Z}_p})$ arising as the p-torsion points in the multiplicative group

$\mathbb{G}_m$, should be changed to q-torsion points of some other geometric objects where q is the cardinal of residue field of F. For example, when F is an imaginary quadratic field, we shall use elliptic curves with complex multiplication. The real difference is that we are changing from $f(T) = x^{p^n} - 1$ to some other (formal) polynomial to express "torsion points". Therefore we need to introduce formal groups or p-divisible groups.

Move to the nonabelian case:

Let G be a reductive group. Via the local Langlands correspondence, we can relate certain representations of G to some homomorphisms of Galois groups (to be more precise: certain classes of homomorphisms to the ${}^LG$). Therefore, like the case for the Artin map, we have the desire to describe the correspondence. In order to achieve this, we need to generalize the above process to the nonabelian cases. The situation is much more complicated because we no longer have uniqueness when passing from objects over residue field to original fields. For example for $\mathrm{GL}_n(F)$, we need to consider moduli spaces of certain elements and the formal objects defined by them. Then we look at rigid fibers of those formal objects: these rigid fibers form a tower, called the Lubin-Tate tower. We consider their ℓ-adic cohomology groups to capture information. For a general reductive group, we need to use Rapoport-Zink spaces or local Shimura varieties. In the process of building these theories, we see the need of more general theory of algebraic geometry to describe the inverse limit of rigid fibers. This motivates us to go to the perfectoid world.

4. General theory

4.1. **σ-linear algebra.** Recall that we require the field k to be algebraically closed of characteristic p, we set $K_0 = W(k)$ and $E \supset F \supset \mathbb{Q}_p$ to be a sequence of finite extensions and $\mathbb{F}$ to be the residue field of F. We use L to denote the completion of maximal unramified extension of F in $\widehat{\overline{K_0}}$, which equals $F.K_0$. The Frobenius automorphism of L/F is denoted by σ.

Definition 4.1. A *σ-L-space* or called F-isocrystal over $\overline{\mathbb{F}}$, is a finite dimensional vector space V over L together with a σ-semilinear bijection $\varphi : V \to V$ (*i.e.* φ is a group homomorphism such that $\varphi(\alpha v) = \sigma(\alpha)\varphi(v)$ for all $\alpha \in L$ and $v \in V$). The dimension of V is called the *height* of σ-L-space (V, φ).

When $F = \mathbb{Q}_p$, the σ-L-spaces are called *F-isocrystals*.

For two σ-L-spaces (V, φ) and (V', φ'), a homomorphism between them $f : V \to V'$ is a L-linear map such that $f(\varphi(v)) = \varphi'(f(v))$ for all $v \in V$.

In the name, F stands for "Frobenius". The F-isocrystals form a $\mathbb{Q}_p$-linear category. Moreover, since we assume k to be algebraically closed, it is a noetherian, artinian semi-simple abelian category, see [RZ] §1.1. Its simple objects are parametrized by elements of $\mathbb{Q}$. For $\lambda \in \mathbb{Q}$, $\lambda = r/s$ with $r, s \in \mathbb{Z}$ and $(r, s) = 1$, it corresponds to the simple object

$$E_\lambda = \left(K_0^s, \begin{pmatrix} 0 & 1 & & \\ & & \ddots & 1 \\ p^r & & & 0 \end{pmatrix} \cdot \sigma \right). \tag{4.1}$$

and $D_\lambda = \mathrm{End}(E_\lambda)$ is a division algebra (defined in [Mil2] §IV Example 1.8), with center $\mathbb{Q}_p$ and invariant $-\lambda$.

For any σ-L-space, we write it as $V = \bigoplus V_\lambda$ for its isotypical decomposition. An σ-L-space is called *isotypic* if and only if there are integers r, s with $s > 0$ and a $\mathcal{O}_L$-lattice M in V such that

$$\varphi^s(M) = p^r M.$$

Definition 4.2. A *filterd isocrystal over L* is a triple $(V, \varphi, \mathcal{F}^\bullet)$ given by an F-isocrystal (V, φ) and a decreasing filtration $\mathcal{F}^\bullet$ on the vector space $V \otimes_{K_0} L$ such that $\mathcal{F}^r = (0)$ and $\mathcal{F}^s = V \otimes_{K_0} L$ for some $r, s \in \mathbb{Z}$.

A subobject $(V', \varphi', \mathcal{F}')$ of $(V, \varphi, \mathcal{F})$ is given by a subvector space V' which is φ-stable such that $V' \otimes_{K_0} L$ is equipped with the induced filtration.

Definition 4.3. A filterd isocrystal $(V, \varphi, \mathcal{F}^\bullet)$ over L is called *weakly admissible* if for every subobject $(V', \varphi', \mathcal{F}'^\bullet)$ we have

$$\sum i \cdot \dim \mathrm{gr}_{\mathcal{F}'}(V' \otimes_{K_0} L) \leqslant \mathrm{ord}_p \det(\varphi'),$$

and when $(V', \varphi', \mathcal{F}'^\bullet) = (V, \varphi, \mathcal{F}^\bullet)$, we have equality.

We can construct a map from the category of finite-dimensional F-rational representations V of G to the category of σ-L-spaces

$$V \mapsto (V_L, \varphi) := (V \otimes_F L, b \cdot (\mathrm{id}_V \otimes \sigma)).$$

4.2. **Kottwitz map and Newton map.**

Definition 4.4. Two elements b, $b' \in G(L)$ are called *σ-conjugated*, denoted by $b \overset{\sigma}{\sim} b'$, if there is a $g \in G(L)$ such that $b' = gb\sigma(g)^{-1}$. Let $[b]$ denote the σ-conjugacy class of b, and $B(G) = B(G, F)$ denote the set of σ-conjugacy classes in $G(L)$.

Remark 1. The set $B(G)$ is independent of the choice of k, i.e. if $k' \subset k$ is algebraically closed, then the resulting $B(G, F)$ are the same. It is proved in [RZ] 1.16.

Recall the composition of canonical homomorphisms:

$$\rho : G_{\text{sc}} \to G_{\text{ss}} \to G$$

Definition 4.5. First assume G splits over F for a maximal split torus $T \subset G$, i.e. G contains a maximal torus $T \cong (\mathbb{G}_m)^n$ over F, for some $n \in \mathbb{N}$, consider the canonical morphism ρ as above. We write $T^{(\text{sc})}$ for $\rho^{-1}(T) \subset G^{\text{sc}}$. Set $\pi_1(G) = \pi_1(G, T) := X_*(T)/\rho_*(T^{(\text{sc})})$. This abelian group is called *the algebraic fundamental group* of G.

Now let G be any (not necessarily split) connected reductive group. By the algebraic fundamental group of G we mean $\pi_1(G_{\overline{F}})$.

This definition is from [Boroi], We have canonical identifications with what was used originally by Kottwitz (cf. [Boroi] proposition 1.10, and [Kot1]):

$$\pi_1(G)_\Gamma = X^*(Z(\widehat{G})^\Gamma)$$

where $\widehat{G}$ is the connected component of ${}^LG$, ${}^LG$ is L-group defined in [Bor] §I.2.

In particular,

$$\begin{aligned} X^*(Z(\widehat{G})^\Gamma) &= \pi_1(G)_\Gamma \\ \mathrm{Hom}(\pi_0(Z(\widehat{G})^\Gamma), \mathbb{C}^\times) &= (\pi_1(G)_\Gamma)_{\text{tors}} \end{aligned} \tag{4.2}$$

Let $T \subset B \subset G_{\overline{F}}$ be a maximal torus and a Borel subgroup defined over $\overline{F}$, then the action of Γ on $X_*(T)$ is defined by

$$\tau \bullet \mu := \mathrm{Int}(g) \circ \tau(\mu), \quad \forall \tau \in \Gamma,\ \forall \mu \in X_*(T) \tag{4.3}$$

where $g \in G(\overline{F})$ satisfies $\mathrm{Int}(g) \circ \tau(T, B) = (T, B)$, and $\tau(\mu)(\cdot) = \tau\mu(\tau^{-1}\cdot)$. It induces an action of Γ on $\pi_1(G)$.

Kottwitz constructed in [Kot4] §7 a group homomorphism

$$\tilde{\kappa}_G : G(L) \to X^*(Z(\widehat{G})^I) = \pi_1(G)_I$$

When G^{der} is simply connected, $\tilde{\kappa}_G$ factors through G^{ab} : $\tilde{\kappa}_G = \tilde{\kappa}_{G^{ab}} \circ p_G$, where $p_G : G \to G^{ab}$ is the natural projection. There is also a homomorphism

$$v_G : G(L) \to \mathrm{Hom}(X_*(Z(\widehat{G})^I, \mathbb{Z})) \tag{4.4}$$

sending $g \in G(L)$ to the homomorphism $\chi \mapsto v_L(\chi(g))$ from $X_*(Z(\widehat{G}))^I = \mathrm{Hom}_L(G, \mathbb{G}_m)$ to $\mathbb{Z}$, where v_L is normalized so that ϖ_L has value 1. From the definition of v_G, we have $v_G = v_{G^{ab}} \circ p_G$ for any G.

Then there is the relation $v_G = q_G \circ \tilde{\kappa}_G$, where q_G is the natural surjective map

$$q_G : \pi_1(G)_I = X^*(Z(\widehat{G})^I) = X^*(Z(\widehat{G}))_I \to \mathrm{Hom}(X_*(Z(\widehat{G}))^I, \mathbb{Z}). \tag{4.5}$$

The kernel of q_G is the torsion subgroup of $X^*(Z(\widehat{G}))_I$. If $\pi_1(G)_I$ is is induced, *i.e.* has a $\mathbb{Z}$-basis permuted by I, q_G is an isomorphism.

We take $H^0(\Gamma,-)$ on both sides of $\tilde{\kappa}_G$ and obtain a homomorphism

$$\lambda_G : G(F) \to X^*(Z(\widehat{G})^I)^{\langle\sigma\rangle}.$$

For $x \in \pi(G)_I$, we denote the image under the natural quotient map $\pi(G)_I \to \pi(G)_\Gamma$ by $\overline{x}$, then $\tilde{\kappa}_G$ induces a map of sets

$$\kappa_G : B(G) \to X^*(Z(\widehat{G})^\Gamma) = \pi_1(G)_\Gamma : \kappa_G([b]) = \overline{\tilde{\kappa}_G(b)} \tag{4.6}$$

where b is any representative of $[b]$.

Definition 4.6. (**Kottwitz map**) Define the *Kottwitz map* $\kappa_G : B(G) \to \pi_1(G)_\Gamma$ as above.

Let $\mathbb{D} := \varprojlim_E \mathrm{Res}_{E/F}\,\mathbb{G}_m$ be the pro-algebraic torus defined over F with character $\mathbb{Q}$. It has character group $\mathbb{Q}$, and there is a canonical projection $\mathbb{D} \to \mathbb{G}_m$ dual to the inclusion of characters $\mathbb{Z} \to \mathbb{Q}$ (*cf.* [DOR] p115 Examples 4.2.1). We put

$$\mathcal{N}(G) := (\mathrm{Int}\,G(L)\backslash \mathrm{Hom}_L(\mathbb{D},G))^{\langle\sigma\rangle} \tag{4.7}$$

where quotient of $\mathrm{Int}\,G(L)$ means modulo equivalent relation defined by conjugacy action $\mathrm{Int}(g) := g(\cdot)g^{-1}$ for $g \in G(L)$. If T is a maximal torus of G with Weyl group W, then

$$\mathcal{N}(G) = (X_*(T)_{\mathbb{Q}}/W)^\Gamma \tag{4.8}$$

where $X_*(T)_{\mathbb{Q}} = X_*(T)\otimes\mathbb{Q}$. The proof follows from [Kot1] and the fact that $\mathcal{N}(T) = X_*(T)^\Gamma\otimes\mathbb{Q}$.

Definition-Proposition 4.7. (**Newton map**) The group $\mathbb{Q}^\times$ acts on the character group $\mathbb{Q}$ of $\mathbb{D}$, thus it acts on $\mathbb{D}$. Let $b \in G(L)$, then there exists a unique element $\nu \in \mathrm{Hom}_L(\mathbb{D},G)$ for which there exist an integer $s > 0$, an element $c \in G(L)$ and a uniformizing element ϖ of F such that:

(i) $s\nu \in \mathrm{Hom}_L(\mathbb{G}_m, G)$, where $s\nu$ denotes the composite $\mathbb{D} \xrightarrow{s} \mathbb{D} \xrightarrow{\nu} G$;
(ii) $\mathrm{Int}(c)\circ\nu$ is defined over the fixed field of σ^s in L;
(iii) $c\cdot b\cdot\sigma(b)\cdot\ldots\cdot\sigma^s(b)\cdot\sigma^s(c)^{-1} = c\cdot(s\nu)(\varpi)\cdot c^{-1}$.

The element ν is called the *slope homomorphism* associated to b.

Furthermore, the map $b \mapsto \nu$ that we may also denote by ν_b or $\nu_{G,b}$ has the following properties:

(a) $\nu_{\sigma(b)} = \sigma(\nu)$.
(b) $gb\sigma(g)^{-1} \mapsto \mathrm{Int}(g)\circ\nu$, $\forall g \in G(L)$.
(c) $\nu_b = \mathrm{Int}(b)\circ\sigma(\nu_b)$.
(d) ν_b is trivial if and only if b is in the image of the map $H^1(F,G) \to B(G)$ (cf. [Kot2] eq.1.8.3). By taking conjugacy classes: $\nu_G([b]) := \overline{\nu}_{G,[b]} = \overline{\nu}_b$, for any $b \in [b]$, we call the map $\nu_G([b])$ *the Newton map* of the group G.

This definition-proposition is subtracted from [Kot2] §4.

Remark 2. (1) (b) shows that the Newton map is well-defined, independent of choice of $b \in [b]$. (c) shows that the image ν is defined over F if, for example, when G is quasi-split.
(2) The morphism ν_b defines a $\mathbb{Q}$-grading on the vector space $V\otimes_F L$ for any F-rational representation of G. The morphism ν_b is characterized by the property that this grading is the slope decomposition of the isocrystal associated to (b,V). The slope λ of V that appear in the slope decomposition is independent of the choice of b in $[b]$.

(3) Note that both κ_G and ν_G are functorial in G, $\nu_{(-)} : B(-) \to \mathcal{N}(-)$ and $\kappa_{(-)} : B(-) \to \pi_1(-)_\Gamma$ are natural transformations of functors from the category of connected reductive algebraic groups to the category of finitely generated discrete Γ-modules.
(4) $\nu_G(b)$ and $\kappa_G(b)$ have the same image in $\pi_1(G) \otimes \mathbb{Q}$. cf. [Kot4] §6.

Proposition 4.8. *For $G = \mathrm{GL}(V)$, where V is a h-dimensional vector space over F, then the set $B(G)$ classifies the σ-L-spaces of height h,* i.e. *there is a bijection*

$$B(G) \xrightarrow{\sim} \{F\text{-}isocrystals\}$$

Proof. For $b \in G(L)$, we associate to it an σ-L-space

$$(V_L, \varphi) := (V \otimes_F L, b \cdot (\mathrm{id}_V \otimes \sigma)).$$

There exist uniquely determined rational numbers

$$\lambda_1 < \lambda_2 < \cdots < \lambda_r$$

and a uniquely determined decomposition

$$V_L = \bigoplus_{i=1}^{r} V_i.$$

into the φ-stable subspaces for which there exist $\mathcal{O}_L$-lattices $M_i \subset V_i$ such that

$$\varphi^{h_i} M_i = \varpi_L^{d_i} M_i, \quad h_i = \dim_L V_i$$

where $d_i = \lambda_i \cdot h_i \in \mathbb{Z}$. The subspace V_i is called the *isotypical component of slope* λ_i. The associated ν_b is equal to

$$\nu_b = \bigoplus_{i=1}^{r} \lambda_i \cdot \mathrm{id}_{V_i}$$

Here $\lambda_i \cdot \mathrm{id}_{V_i}$ denotes the composition

$$\mathbb{D} \xrightarrow{\lambda_i} \mathbb{G}_m \subset \mathrm{GL}(V)$$

In this case the map $\overline{\nu}_G : B(G) \to \mathcal{N}(G)$ is injective, following from Dieudonné-Manin classification of σ-L-spaces, [Kot2] §3. □

Definition 4.9. Associate to $b \in G(L)$ there is a functor

$$J_b(R) := \{g \in G(R \otimes_F L) \mid b = gb\sigma(g)^{-1}\}.$$

Let J_b be an algebraic group that represents this functor, it is called *the σ-centralizer group.*

The fact that functor J_b is representable by a smooth affine group scheme is proved in [RZ] Proposition 1.12.. Note that $J_b = J_{gb\sigma(g)^{-1}}$ for any $g \in G(L)$.

Definition 4.10. A class $[b] \in B(G)$ is called *basic*, if the conjugacy class $\nu_G([b])$ consists of central morphisms, i.e. its image is in $Z(G)$. Denote the set of basic classes by $B(G)_{\mathrm{basic}}$.

The conjugacy class $[b]$ is basic is equivalent to J_b is an inner form of G ([Kot2] §5).

Proposition 4.11. *The map*

$$\kappa_G \times \nu_G : B(G) \to \pi_1(G) \times (X_*(G)_{\mathbb{Q}}/W)^\Gamma$$

is injective.

This proposition is proved in [Kot4] §4.13.

4.3. **Rapoport-Zink spaces.** Before we introduce local Shimura varieties, we first have a look at Rapoport-Zink spaces because we know more about them and they serve as important examples of local Shimura varieties. This part together with local Shimura varieties should be thought as some general machine, so they look abstract. Later in Example parts 5, we will see hints to various conditions of this section.

Definition 4.12. *A simple rational RZ datum* in *EL case* is a tuple $\mathcal{D}$ of the form $\mathcal{D} = (F, B, V, \{\mu\}, [b])$, where

(1) F is a finite field extension of $\mathbb{Q}_p$,
(2) B is a central division algebra over F,
(3) V is a finite dimensional B-module,
(4) $\{\mu\}$ is a conjugacy class of *minuscule* cocharacters $\mu : \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$, *i.e.* $\langle \mu, \alpha \rangle = \pm 1$ or 0 for all $\alpha \in \Phi(G, B, T)$,
(5) $[b] \in A(G, \{\mu\})$, where $G := \mathrm{GL}_B(V)$ is an algebraic group over $\mathbb{Q}_p$.

And the following additional condition is satisfied: For $\mu \in \{\mu\}$ consider the decomposition of $V \otimes \overline{\mathbb{Q}}_p$ into weight spaces, the only weights occur are 0 and 1.

A simple integral RZ datum $\mathcal{D}_{\mathbb{Z}_p}$ in the EL case consists, in addition to data $\mathcal{D}$, of a maximal order $\mathcal{O}_B$ in B and an $\mathcal{O}_B$-stable lattice Λ in V. This induces an integral model $\mathcal{G}$ of G over $\mathbb{Z}_p$, namely $\mathcal{G} = \mathrm{GL}_{\mathcal{O}_B}(\Lambda)$ as a group scheme over $\mathbb{Z}_p$.

Definition 4.13. For this case, we consider $p \neq 2$. *A simple rational RZ datum* in the PEL case is a tuple $\mathcal{D} = (F, B, V, (\ ,\ ), *, \{\mu\}, [b])$ where

(1) F, B and V are as in EL case,
(2) $(\ ,\ )$ is a nondegenerate alternating $\mathbb{Q}_p$-bilinear form on V,
(3) $*$ is an involution on B satisfying

$$(xv, w) = (v, x^* w), \text{ for all } v, w \in V, \text{ and all } x \in B,$$

(4) $\{\mu\}$ is a conjugacy class of minuscule cocharacters $\mu : \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$, where G is the algebraic group over $\mathbb{Q}_p$ defined by

$$\begin{aligned} G(R) = \{ g \in \mathrm{GL}_{B \otimes_{\mathbb{Q}_p} R}(V \otimes_{\mathbb{Q}_p} R) \mid & \text{ there is} \\ & c(g) \in R^\times \text{ such that } (gv_1, gv_2) = c(g)(v_1, v_2), \text{ for all } v_1,\ v_2 \in V \otimes_{\mathbb{Q}_p} R \}, \end{aligned} \tag{4.9}$$

(5) $[b] \in A(G, \{\mu\})$.

And the following additional condition is satisfied:

(a) For $\mu \in \{\mu\}$ consider the decomposition of $V \otimes \overline{\mathbb{Q}}_p$ into weight spaces, the only weights occur are 0 and 1.
(b) We require for any $\mu \in \{\mu\}$, the composition

$$\mathbb{G}_{m,\overline{\mathbb{Q}}_p} \xrightarrow{\mu} G_{\overline{\mathbb{Q}}_p} \xrightarrow{c} \mathbb{G}_{m,\overline{\mathbb{Q}}_p}$$

is the identity. The later morphism denotes the multiplier $c : G \to \mathbb{G}_m$.

A simple integral RZ datum $\mathcal{D}_{\mathbb{Z}_p}$ in the PEL case consists in addition to data $\mathcal{D}$, of a maximal order $\mathcal{O}_B$ in B that is stable under involution $*$, and an $\mathcal{O}_B$-stable lattice Λ in V such that $\varpi\Lambda \subset \Lambda^\vee \subset \Lambda$. Here ϖ denotes the uniformizer in $\mathcal{O}_B$, and $\Lambda^\vee$ denotes the dual integral lattices with respect to $(\ ,\ )$. This induces an integral model $\mathcal{G}$ of G over $\mathbb{Z}_p$ with $\mathcal{G}(\mathbb{Z}_p) = G(\mathbb{Q}_p) \bigcap \mathrm{GL}_{\mathcal{O}_B}(\Lambda)$.

In the rest of this part, We will abbreviate J for J_b, and by writing b, we always mean a representative b in a fixed $[b]$. Let E be the definition field of $\{\mu\}$ and $\mathcal{O}$ be the ring of integers of $\mathcal{O}_{\breve{E}}$, let $\mathcal{O}_E$ denote the ring of integers of E.

Let $\mathcal{N}ilp_{\mathcal{O}}$ denote the category of $\mathcal{O}$-schemes S on which p is locally nilpotent. For $S \in \mathcal{N}ilp_{\mathcal{O}}$ we denote by $\overline{S}$ the closed subscheme defined by $p\mathcal{O}_S$.

Definition 4.14. An abelian fppf sheaf G(see [Vistoli] §2.3) is said to be a *p-divisible group* if the following conditions are satisfied:

(i) $G \xrightarrow{\times p} G$ is surjective.
(ii) $G[p^m] := \mathrm{Ker}(G \xrightarrow{\times p^m} G)$ is represented by a finite locally free group scheme over S.
(iii) $G = \varinjlim_m G[p^m]$.

A morphism $f : G' \to G$ of p-divisible group is called *an isogeny* if its kernel is a finite group scheme. A *quasi-isogeny* between p-divisible groups is an isogeny multiplied by ϖ^{-n} for some $n \in \mathbb{N}$.

Let $\overline{\mathbb{F}}_{\breve{E}}$ be the residue field of $\mathcal{O}$, and denote it by $\overline{\mathbb{F}}'$ for brevity. We fix a pair $(\mathbb{X}, \rho_{\mathbb{X}})$ as framing object.

Definition 4.15. We define pairs (X, ρ) where X is a p-divisible group over $S \in \mathcal{N}ilp_{\mathcal{O}}$ and $\rho : \mathcal{O}_B \to \mathrm{End}(X)$ is an action of $\mathcal{O}_B$ on X. We require the *Kottwitz condition* associated to $\{\mu\}$, *i.e.* the equality of characteristic polynomials

$$\mathrm{char}(\rho(b)|_{\mathrm{Lie}\,X}, T) = \mathrm{char}(b|_{V_0}, T),\ \forall b \in \mathcal{O}_B \tag{4.10}$$

where V_0 is the weight 0 subspace under weight space decomposition associated to any $\mu \in \{\mu\}$. We require the rational Dieudonné module (*cf.* [Wei] Lecture 1) of $\mathbb{X}$ with its action by B and its Frobenius endomorphism is isomorphic to $(V \otimes_F L, b \cdot (\mathrm{id} \otimes \sigma))$ with $b \in [b]$ fixed.

Then we consider the set-valued functor

$$\begin{aligned} M_{\mathcal{D}_{\mathbb{Z}_p}} : \mathcal{N}ilp_{\mathcal{O}} &\to (\mathrm{Sets}) \\ S &\mapsto \{(X, \rho, \iota)\} / \cong \end{aligned} \tag{4.11}$$

where

$$\left\{(X, \rho, \iota) \mid (X, \rho) \text{ is as above, with a } \mathcal{O}_B\text{-linear quasi-isogeny } \iota : X \times_S \overline{S} \to \mathbb{X} \times_{\mathrm{Spec}\,\overline{\mathbb{F}}'} \overline{S}\right\} \tag{4.12}$$

We can similarly define such functor for PEL case, see [RV] §4.6.

Theorem 4.16. *Let $\mathcal{D}_{\mathbb{Z}_p}$ be integral RZ data of type EL or PEL. The functor $M_{\mathcal{D}_{\mathbb{Z}_p}}$ on $\mathcal{N}ilp_{\mathcal{O}}$ is representable by a formal scheme, locally formally of finite type and separated over* $\mathrm{Spf}\,\mathcal{O}$.

Proof. [RZ] Theorem 2.16. □

Fix a simple integral RZ datum, let $M = M_{\mathcal{D}_{\mathbb{Z}_p}}$ be the corresponding formal scheme over $\mathrm{Spf}\,\mathcal{O}$ and pass to its generic fibers using [Hub] §4: $\mathcal{M} := M^{\mathrm{rig}}$. Let $\mathcal{T}$ be the local system over $\mathcal{M}$ defined by the p-adic Tate module of the universal p-divisible group on M, together with the $\mathcal{O}_B$ action and polarization pairing (in the PEL case). Set $\mathcal{V} = \mathcal{T} \otimes \mathbb{Q}_p$. Let $K \subset \mathcal{G}(\mathbb{Z}_p)$ be a subgroup of finite index, we associate to K the rigid space $\mathcal{M}^K = \mathcal{M}^K_{\mathcal{D}_{\mathbb{Z}_p}}$ classifying K-level structure of $\mathcal{V}$

$$\mathcal{T} \cong \Lambda \bmod K \tag{4.13}$$

see [Far1] Definition 2.3.17. Then $\mathcal{M}^K$ is a finite étale covering of $M^{\mathrm{rig}}_{\mathcal{D}_{\mathbb{Z}_p}}$ and for $K_0 = \mathcal{G}(\mathbb{Z}_p)$, we define $\mathcal{M}^{K_0} := M^{\mathrm{rig}}_{\mathcal{D}_{\mathbb{Z}_p}}$.

There is a smooth projective variety $\mathcal{F} = \mathcal{F}(G, \{\mu\})$ over E where E is the definition field of μ, whose points over $\overline{F}$ correspond to the par-equivalence classes of elements in $\{\mu\}$. The variety $\mathcal{F}$ is homogeneous under G_E, and is a generalized flag variety for G_E.

Set

$$\breve{\mathcal{F}}^{\mathrm{rig}} := \breve{\mathcal{F}}(G, \{\mu\})^{\mathrm{rig}} = (\mathcal{F} \times_{\mathrm{Spec}\, E} \mathrm{Spec}\, \breve{E})^{\mathrm{rig}}.$$

Definition 4.17. We call a pair (b, μ) consisting of an element $b \in G(L)$ and a cocharacter μ of G defined over a finite extension K of L *a weakly admissible pair* in G if for any F-rational representation V of G, the filtered σ-F-space $\mathcal{I}(V) = (V \otimes_F L, b\sigma, \mathcal{F}^{\bullet}_{\mu})$ is weakly admissible.

Definition 4.18. The set of elements $\mu \in \breve{\mathcal{F}}(\overline{F})$ such that (b, μ) is *a weakly admissible pair* for G forms an admissible open subset of $\breve{\mathcal{F}}^{\mathrm{rig}}$, called *the nonarchimedean period domain associated with* $(G, b, \{\mu\})$ and denoted by $\mathcal{F}^{wa} = \breve{\mathcal{F}}^{\mathrm{rig}}(G, b, \{\mu\})^{\mathrm{wa}}$, *cf.* [RZ] Definition 1.35.

Let $(X_{\mathrm{univ}}, \iota_{\mathrm{univ}})$ be the universal p-divisible group over $\mathcal{M}$, with additional structure and equipped with the universal quasi-isogeny. Then ι induces an isomorphism

$$V \otimes_{\mathbb{Q}_p} \mathcal{O}_{\mathcal{M}} \xrightarrow{\sim} M(X_{\mathrm{univ}}) \otimes_{\mathcal{O}_M} \mathcal{O}_{\mathcal{M}}$$

where $M(X_{\mathrm{univ}})$ denotes the Lie algebra of the universal vector extension of X_{univ} and V the rational Tate module of X. The surjection $M(X) \to \mathrm{Lie}\, X$ thus yields a filtration on $V \otimes_{\mathbb{Q}_p} \mathcal{O}_{\mathcal{M}}$ which corresponds to a morphism $\breve{\pi} : \mathcal{M} \to \breve{\mathcal{F}}^{\mathrm{rig}}$ which factors through $\mathcal{F}^{\mathrm{wa}}$. The period morphism extends to a compatible system of morphisms

$$\breve{\pi}^K : \mathcal{M}^K_{\mathcal{D}} \to \breve{\mathcal{F}}^{\mathrm{rig}}, \tag{4.14}$$

which factors through $\mathcal{F}^{wa}$ and but not compatible with *Weil descent data*, see [RZ] Definition 3.45. Therefore we need to modify this by introducing another map in the next subsection.

4.4. **Local Shimura varieties and their cohomology.** In this section, we assume that G is quasi-split and split over an unramified extension E of F. We assume from now on that ν_b is chosen (in a unique way) so that it represents $\nu_G([b]) \in (X_*(T)_{\mathbb{Q}})^{\Gamma}_{\mathrm{dom}}$. On $X_*(T)_{\mathbb{Q}}$ we consider the order $\leqslant$ given by $v \leqslant v'$ if and only if $v' - v$ is a nonnegative $\mathbb{Q}$-linear combination of positive relative coroots. Let $(X_*(T)_{\mathbb{Q}})_{\mathrm{dom}}$ denote the set of cocharacters of T which are dominant with respect to B.

For any conjugacy class of $\{\mu\}$ of $X_*(T)$, denote by μ_{dom} the unique element of $\{\mu\}$ in $X_*(T)_{\mathrm{dom}}$.

For any dominant element $\mu \in X_*(T)$, let Γ_μ be the stabilizer of $\mu \in \Gamma$, it has finite index as $X_*(T)$ is a discrete Γ-module. Define $\overline{\mu}$:

$$\overline{\mu} = [\Gamma : \Gamma_\mu]^{-1} \sum_{\tau \in \Gamma/\Gamma_\mu} \tau(\mu) \in (X_*(T)_{\mathbb{Q}})^{\Gamma}_{\mathrm{dom}}$$

We know from the definition of action (4.3) that τ action preserves the set of roots Φ and the resulting $\overline{\mu}$ is just "taking the average", thus $\overline{\mu}$ is in $(X_*(T)_{\mathbb{Q}})^{\Gamma}_{\mathrm{dom}}$.

Let $b \in G(F)$ and $\mu \in X_*(T)$ be such that $b \in K\mu(\varpi_F)K$ (by Cartan decomposition) for some hyperspecial subgroup K of $G(F)$ that fixes a vertex in the apartment for $T(L)$. Then $\kappa_G(b)$ is the image of μ under the canonical projection appearing in definition 4.5 $X_*(T) \twoheadrightarrow \pi_1(G)_{\Gamma}$. We define $\mu^{\sharp}$ as the image of μ under this canonical projection.

Definition 4.19. If $\{\mu\}$ is a conjugacy class of cocharacters over $\overline{F}$, an element $[b] \in B(G)$ is called *acceptable* with respect to $\{\mu\}$ if $\nu_G([b]) \leqslant \overline{\mu}_{\mathrm{dom}}$. Denote by $A(G, \{\mu\})$ the subset of acceptable elements of $B(G)$.

Definition 4.20. A class $[b] \in A(G, \{\mu\})$, $[b]$ is called *neutral* if $\kappa_G([b]) = \mu^\sharp$, denote by $B(G, \{\mu\})$ the set of acceptable neutral conjugacy classes.

Remark 3. There is an interpretation of the above conditions in terms of the Hodge polygon and the Newton polygon when $G = \mathrm{GL}_n$: $\overline{\mu}$ is called the *generalized Hodge polygon*.

$\kappa_G(b) = \mu^\sharp \Leftrightarrow$ The Hodge polygon and the Newton polygon have the same endpoints.

$\nu_G([b]) \leqslant \overline{\mu} \Leftrightarrow$ The Hodge polygon lies above the Newton polygon.

$B(G, \{\mu\})$ and $A(G, \{\mu\})$ can be defined more generally when G is no longer assumed to be quasi-split ([Kot4], 6.2). They are nonempty finite sets.

Definition 4.21. A *local Shimura datum* over F is a triple $(G, [b], \{\mu\})$ consisting of the following data:

(a) G is a reductive group over F;
(b) $[b] \in B(G)$ is a σ-conjugacy class;
(c) $\{\mu\}$ is a geometric conjugacy class of cocharacters, *i.e.* $\mu : \mathbb{G}_{m,\overline{F}} \to G_{\overline{F}}$.

such that:

(1) $\{\mu\}$ is minuscule,
(2) $[b] \in B(G, \{\mu\})$.

To a local Shimura datum is associated:

(1) the reflex field $E = E(G, \{\mu\})$ which is the definition field of μ inside $\overline{F}$;
(2) the algebraic group J_b.

The (conjectural) existence of Local Shimura Varieties We conjecture that for a given local Shimura datum $(G, [b], \{\mu\})$ over F, there exists a tower of rigid-analytic spaces $\{\mathcal{M}^K\}_K$ over $\mathrm{Sp}\,\breve{E}$, where K ranges over all open compact subgroups of $G(F)$, with the following properties:

(i) each $\mathcal{M}^K$ is equipped with an action of $J(F)$.
(ii) the group $G(F)$ operates on the tower as a group of Hecke correspondences, for definition see [RZ] Definition 4.57.
(iii) the tower is equipped with a Weil descent datum down to E,
(iv) there exists a compatible system of étale and partially proper *period morphism(s)* $\breve{\pi}^K : \mathcal{M}^K \to \breve{\mathcal{F}}(G, b, \{\mu\})^{\mathrm{wa}}$ that is equivariant for the action of $J(F)$ and which is the first component of a $J(F) \times G(F)$-equivariant morphism of towers of rigid-analytic spaces

$$(\breve{\pi}^K, \kappa^K) : \mathcal{M}^K \to \mathcal{F}(G, b, \{\mu\})^{\mathrm{wa}} \times \Delta$$

compatible with the Weil descent data. Here Δ is the dual abelian group of $X^*(G_{ab})^\Gamma$, $\breve{\pi}^K$ is defined in (4.14) and $\kappa^K : \mathcal{M}^K \to \Delta$ is defined in [RZ] 3.52.

Now we can define the ℓ-adic cohomology of $\mathcal{M}^K$ with compact support (recall that ℓ is a prime such that $\ell \neq p$), for each $i \in \mathbb{N}$:

$$H_c^i(\mathcal{M}^K) := H_c^i(\mathcal{M}^K \times_{\breve{E}} \widehat{\overline{E}}, \overline{\mathbb{Q}}_\ell) := \left(\left(\varprojlim_r H_c^i(\mathcal{M}^K \otimes_{\breve{E}} \widehat{\overline{E}}, \mathbb{Z}/\ell^r\mathbb{Z}) \right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell) \right) \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell$$

It is equipped with a smooth action of $J(F)$ and a continuous action of the inertia group $I_E = \mathrm{Gal}(\widehat{\overline{E}}/\breve{E})$ by functoriality. Due to the Weil descent datum from $\breve{E}$ to E, the action of I_E can be extended to an action of W_E. Then $H_c^i(\mathcal{M}^K)$ is a finitely generated $J(F)$-module, proved in [Far1] Proposition 4.4.13.

For every admissible representation ρ of $J(F)$, and every compact open subgroup K, indexes i, $j \in \mathbb{N}$, we can define the $\overline{\mathbb{Q}}_\ell$-vector space:

$$H^{i,j}(\mathcal{M}^K)[\rho] = \mathrm{Ext}^j_{J(F)}(H^i_c(\mathcal{M}^K), \rho)$$

which is finite dimensional over $\overline{\mathbb{Q}}_\ell$ and vanishes for $j > \mathrm{rk}_{ss}(J)$, *i.e.* the semisimple rank of J.

Further, we define:

$$H^{i,j}((G,[b],\{\mu\}))[\rho] := \varinjlim_{K} \mathrm{Ext}^j_{J(F)}(H^i_c(\mathcal{M}^K), \rho).$$

This is an admissible $G(F)$-module and a continuous W_E-module, and it vanishes for almost all $i, j \in \mathbb{N}$.

We denote by $\mathrm{Groth}(G(F) \times W_E)$ the Grothendieck group of the category of $(G(F) \times W_E)$-modules over $\overline{\mathbb{Q}}_\ell$ that are admissible as $G(F)$-module and continuous as W_E-modules. We define the alternating sum in $\mathrm{Groth}(G(F) \times W_E)$, modified by a Galois twist, let $d = \dim \mathcal{M}$:

$$H^\bullet((G,[b],\{\mu\}))[\rho] = \sum_{i,j \geqslant 0} (-1)^{i+j} H^{i,j}((G,[b],\{\mu\}))[\rho](-d) \tag{4.15}$$

4.5. **Summary of local Langlands correspondence.** Let us introduce some notations from local Langlands correspondence: Let W'_F be the Weil-Deligne group of F, for our concerned case that F is nonarchimedean, is $W_F \times \mathrm{SL}_2(\mathbb{C})$. We denote by ${}^LG$ the L-group of G, and $\widehat{G} := ({}^LG)^0$. We consider the tempered irreducible admissible representations of $G(F)$, defined in [Bor] §10, whose set of equivalence classes is denoted by $\Pi(G)$. We denote by $\varPhi(G)$ the set of (tempered) admissible L-homomorphisms $\varphi : W'_F \to {}^LG$ modulo equivalence up to $\mathrm{Int}(\widehat{G})$ action, and elements of $\varPhi(G)$ are call *L-parameters* or *Langlands parameters*, see [Bor] §8.2. For $\varphi \in \varPhi(G)$, we consider $S_\varphi := \mathrm{Cent}_{\widehat{G}}(\varphi)$ the centralizer of image of φ in $\widehat{G}$. φ is called a *discrete* L-parameter if the image of φ is not contained in any proper Levi subgroup of ${}^LG$, which is equivariant to $S^0_\varphi \subset Z(\widehat{G})^\Gamma$. The local Langlands conjecture for reductive groups ([Kal] §1.1 Conjecture A) tells us: there is a natural surjection $\mathrm{LLC} : \Pi(G) \to \varPhi(G)$ with finite fibers, or called *L-packet*, denoted by $\Pi_\varphi(G)$ for each $\varphi \in \varPhi(G)$. φ is called *elliptic* if the restriction of φ to $\mathrm{SL}_2(\mathbb{C})$ is trivial. We require all φ for our discussion to be elliptic.

When $G = \mathrm{GL}_n$, it means that φ corresponds to a *supercuspidal* representation under local Langlands correspondence [Ren].

4.6. **The statement of Kottwitz Conjecture.** We set $\lambda_b = \kappa_G([b])$, $\lambda_{b*} = \kappa_G([b^*])$. In order to state the Kottwitz Conjecture, we make the following assumptions throughout this section.

Assumption 4.22. :

(1) b is basic, i.e. J is an inner form of G.
(2) G is a *B-inner twist* of the quasi-split form G^* of G, i.e., there exists $b^* \in G^*(L)$ such that G is isomorphic to J_{b*} of G^*.

Remark 4. (i) The assumption (2) is satisfied if G has connected center.
(ii) If G is quasi-split, and $\lambda_{b*} = 0$.

Recall that $\pi_1(G)_\Gamma = X^*(Z(\widehat{G})^\Gamma)$. From Kaletha's refined local Langlands correspondence [Kal] §1.4, there are identifications of L-packets for G and J_b:

$$\begin{aligned} \Pi_\varphi(G) &\xrightarrow{\sim} \{\text{irreducible algebraic representations } \tau \text{ of } S_\varphi \mid \tau|_{Z(\widehat{G})^\Gamma} = \lambda_{b*}\} \\ \pi &\mapsto \tau_\pi \\ \Pi_\varphi(J) &\xrightarrow{\sim} \{\text{irreducible algebraic representations } \tau \text{ of } S_\varphi \mid \tau|_{Z(\widehat{G})^\Gamma} = \lambda_{b*} + \lambda_b\} \\ \rho &\mapsto \tau_\rho \end{aligned} \tag{4.16}$$

Definition-Proposition 4.23. For any $\mu \in X_*(G)$, there exists a representation $r_{\{\mu\}}$ of ${}^LG$, unique up to isomorphism, satisfying the following two properties:

(a) As a $\widehat{G}$ representation, $r_{\{\mu\}}$ is irreducible with highest weight μ.

(b) Let y be a choice of splitting of $\widehat{G}$ and assume that it is fixed by Γ, then the subgroup W_F of ${}^LG$ acts trivially on the highest weight space of $r_{\{\mu\}}$ corresponding to y.

For the proof, see [Kot1] lemma 2.1.2. The identification $X_*(Z(G)) = X^*(\widehat{G})$ says that we can associate to μ a character of $\widehat{G}$, the irreducible representation generated by μ under $\mathrm{Int}(\widehat{G})$ action is what we need. By (4.7) and (4.8), μ determines an orbit of the Weyl group of $(\widehat{G}, \widehat{T})$ acting on $X^*(\widehat{T})$, thus it makes sense to say $r_{\{\mu\}}$ is an irreducible representation.

CONJECTURE 4.1. (**Kottwitz Conjecture**). Under the assumptions on $(G, b, \{\mu\})$ above, let φ be a discrete Langlands parameter for G. Denote by φ_E the restriction of φ to W_E. Recall that $r_{\{\mu\}}$ is the representation of ${}^LG$ defined by $\{\mu\}$, $\tau_\pi^\vee$ is the contragredient representation of τ_π. See $r_{\{\mu\}} \circ \varphi_E$ as a representation of $S_\phi \times W_E$ via

$$(r_{\{\mu\}} \circ \varphi_E)(s, w) = r_{\{\mu\}}(s \cdot \varphi_E(w)).$$

Then, for $\rho \in \Pi_\varphi(J)$, we have an equality in $\mathrm{Groth}(G(F) \times W_E)$:

$$H^\bullet((G, [b], \{\mu\}))[\rho] = (-1)^d \sum_{\pi \in \Pi_\varphi(G)} \pi \boxtimes \mathrm{Hom}_{S_\varphi}(\check{\tau}_\pi \otimes \tau_\rho, r_\mu \circ \varphi_E)(-\frac{d}{2})$$

where $d = \dim \mathcal{M}$.

Remark 5. M.Harris and R.Taylor have proved the GL_n case with $\mu = (1, 0, \ldots, 0)$ [HT], S.W.Shin has proved it for $\mathrm{Res}_{F/\mathbb{Q}_p}(\mathrm{GL}_n)$, $F/\mathbb{Q}_p$ unramified case, [Shi]. M.Strauch considers the weakened version by ignoring the W_E action, in the Lubin-Tate case for an arbitrary $F/\mathbb{Q}_p$ [Str2].

Remark 6. (i) If φ is a discrete character Langlands parameter, $\mathrm{Ext}^i_{J(F)}(H^i_c(\mathcal{M}^{K'}), \rho) = 0$, $j > 0$, [Ren] VI 3.6. proposition.

(ii) In many cases, studied in [Dat1], we have $\mathrm{Hom}_{J(F)}(H^i_c(\mathcal{M}^K), \rho) = 0$, $i \neq d$.

5. Examples

5.1. **Kottwitz conjecture for torus.** Assume that $G = T$ is a torus over F, for any $\mu : \mathbb{G}_m \to T$, the conjugacy class $\{\mu\}$ consists of only one element as T is commutative. $B(G, \{\mu\})$ consists of a single element, $A(T, \{\mu\}) = (X_*(T)_\Gamma)_{\mathrm{tor}}$, $J_b = T$ since T is commutative. We note that $\pi_1(T) = X_*(T)$, the Kottwitz map is explicitly constructed in [Kot4] §7, λ_b determines ν_b as is shown in [RV] Example 2.2. $r_{\{\mu\}}$ is just μ and $\nu_b = \mu$. We have $\mathcal{M}(T, [b], \mu)^K = T(F)/K$. All τ, π appearing in (4.1) are just characters. The cocharacter μ defines a homomorphism of tori:

$$N_\mu : \mathrm{Res}_{E/F}\,\mathbb{G}_m \xrightarrow{\mathrm{Res}_{E/F}\,\mu} T_E \xrightarrow{N_{E/F}} T$$

where $N_{E/F}$ denotes the norm map. Let

$$\mathrm{Art}_E : E^\times \to \mathrm{Gal}(\overline{E}/E)^{\mathrm{ab}}$$

be the Artin map normalized by $\varpi \mapsto \mathrm{Frob}$ in $\mathrm{Gal}(\overline{E}/E)$, where Frob means the arithmetic Frobenius morphism in $\mathrm{Gal}(\overline{E}/E)$. Let $\chi_\mu : I_E \to T(F)$ be the following composition map:

$$\begin{aligned} I_E &\to \mathcal{O}_E^\times \xrightarrow{N_\mu} T(F) \\ \gamma &\mapsto N_\mu(\mathrm{Art}^{-1}(\gamma|_{E^{\mathrm{ab}}})) \end{aligned} \tag{5.1}$$

The action of I_E on $T(F)/K$ is given by

$$\forall x \in T(F), \quad \gamma(xK) = \chi_\mu(\gamma)xK$$

For varying K, these $T(F)/K$ form a tower of rigid-analytic spaces over $\mathrm{Sp}(\breve{E})$, the action of $J(F) \times T(F)$ on K is by $(a, b)xK = abxK$, this action maps each element of the tower to itself.

The simplifications in the remarks 6 hold and note that $d = 0$. We recall that the action of $J \times W$ is smooth, thus

$$\varinjlim_K H_c^0(\mathcal{M}^K) = \varinjlim_K H_c^0((T(F)/K) \times_{\mathrm{Sp}\,\breve{E}} \mathrm{Sp}\,\widehat{\overline{E}},, \overline{\mathbb{Q}}_\ell) = \varinjlim_K C_c^\infty(T(F)/K) = C_c^\infty(T(F))$$

Since T is compact, by Peter-Weyl theorem [Sep] Theorem 3.24, $C_c^\infty(T(F)) = C^\infty(T(F)) = \bigoplus_{\tau \in \widehat{T}(F)} \tau \otimes \check{\tau}$. Hence on the left hand side of (4.1) becomes:

$$H^\bullet((T, [b], \{\mu\}))[\rho] = \mathrm{Hom}_{T(F)}(C^\infty(T(F)), \rho) = \mathrm{Hom}_{T(F)}(\bigoplus_{\tau \in \widehat{T}(F)} \tau \otimes \check{\tau}, \rho) = \rho$$

We have $S_\varphi = \widehat{T}$ and:

$$\Pi_\varphi(T) \xrightarrow{\sim} \{\tau \text{ characters of } \widehat{T} \mid \tau|_{\widehat{T}^\Gamma} = 0\}$$

$$\Pi_\varphi(J) \xrightarrow{\sim} \{\tau \text{ characters of } \widehat{T} \mid \tau|_{\widehat{T}^\Gamma} = \lambda_b\}$$

In the torus case, the right hand side of (4.1):

$$\mathrm{Hom}_{\widehat{T}}(\tau_\rho, r_\mu \circ \varphi_E) = \mathrm{Hom}_{\widehat{T}}(\rho, \mu) = \rho$$

5.2. **Lubin-Tate formal group laws.** Recall from local Langlands correspondence for $\mathrm{GL}_1(F)$, we have $F^\times \xrightarrow[\mathrm{Art}_F]{\sim} W_F^{ab}$. Lubin-Tate theory understands Art_F by 1-dimensional $\mathcal{O}_F$-module.

We will write objects after reduction, *i.e.* base change to residue field, in math bold type, for example $\mathbb{X}_{\mathbb{F}} := X \otimes_{\mathcal{O}_F} (\mathcal{O}_F/\varpi\mathcal{O}_F)$ where X is defined over $\mathcal{O}_F$.

Definition 5.1. A *one-dimensional commutative formal group law* X over a commutative ring A is a power series $X(\cdot,\cdot) \in A[[S,T]]$ satisfying the properties:

(1) $X(S,T) = S + T+$ higher terms;
(2) $X(S,T) = X(T,S)$;
(3) $X((S,T),U) = X(S,X(T,U))$;
(4) There is a $g(T) \in A[[T]]$ with $X(T,g(T)) = 0$.

Definition 5.2. A 1-dimensional formal $\mathcal{O}_F$-module law over a commutative ring A is a formal group law X over A together with a family of power series $[a]_X$ for $a \in \mathcal{O}_F$ that represent a homomorphism $\mathcal{O}_F \to \mathrm{End}_A(X)$ such that $[a]_X(T) = aT+$ higher terms.

If we consider $A = \overline{\mathbb{F}}$, the residue field of maximal unramified extension F^{un}, then the homomorphism $\mathcal{O}_F \to \mathrm{End}_{\mathbb{F}}(\mathbb{X})$ is just the reduction map $\mathcal{O}_F \to \mathcal{O}_F/\varpi_F\mathcal{O}_F = \mathbb{F} \subset \overline{\mathbb{F}}$.

Let $f(T) \in \mathcal{O}_{\breve{F}}[[T]]$ be any power series satisfying:

(1) $f(T) = \varpi T+$ higher terms,
(2) $f(T) \equiv T^q (\mathrm{mod}\, \varpi)$.

Theorem 5.3. *There exists a unique* 1*-dimensional formal* $\mathcal{O}_F$*-module law* X_f *over* $\mathcal{O}_{\breve{F}}$ *for which* $[\varpi]_{X_f}(T) = f(T)$ *and of height* 1*,* i.e. $[\varpi]_X(T) \cong T^q \mod \varpi$*. Furthermore, if* g *is another power series satisfying the two criteria above, then* X_f *and* X_g *are isomorphic.*

Proof. See [Mil2] Corollary 2.16. □

We define the following objects:

$X[\varpi^m](\overline{\mathcal{O}_{\breve{F}}}) := \{x \in \overline{\mathcal{O}_{\breve{F}}},\ [\varpi^m]_X(x) = 0\}$, abbreviated as $X[\varpi^m]$. Using some facts from commutative algebra, we can prove that $X[\varpi^m]$ is a free $\mathcal{O}_F/\varpi^m\mathcal{O}_F$-module of rank 1. $T_\varpi(X) := \varprojlim_m X[\varpi^m]$, $\breve{F}_m := \breve{F}(X[\varpi^m])$, $\breve{F}_\infty := \bigcup_m \breve{F}_m$.

$\mathrm{Gal}(\breve{F}_m/\breve{F})$ action on $X[\varpi^m] \cong \mathcal{O}_F/\varpi^m\mathcal{O}_F$ naturally thus it induces an isomorphism: $\mathrm{Gal}(\breve{F}_m/\breve{F}) \xrightarrow[\gamma_m]{\sim} (\mathcal{O}_F/\varpi^m\mathcal{O}_F)^\times$. Pass to inverse limit,

$$\varprojlim_m \gamma_m : \mathrm{Gal}(\breve{F}_\infty/\breve{F}) \xrightarrow{\sim} \mathcal{O}_F^\times.$$

From local class field theory, and the identification $\mathrm{Gal}(\breve{F}_\infty/\breve{F}) \cong \mathrm{Gal}(F^{ab}/F^{un}) = I_F^{ab}$ where I_F^{ab} denotes the abelianized inertia group, we deduce $\varprojlim \gamma_m = \mathrm{Art}_F^{-1}|_{I_F^{ab}}$.

5.3. **Kottwitz conjecture for** GL_1**.** In the subsection above, let $h = 1$, $\mathcal{M}$ constructed there has dimension 0. The p-divisible group $\mathbb{X}_0$ is just the multiplicative group, and since any lift of $\mathbb{X}_0$ is also the multiplicative group, as is shown in Theorem 5.3. Therefore $\mathcal{M}$ is the discrete sets with level structure $\alpha : F \xrightarrow{\sim} \varprojlim_n \mu_{q^n}(F) \otimes_{\mathcal{O}_F} F \cong F$, where $\mu_{q^n}(F) := \{x \in F \mid x^{q^n} = 1\}$. $G = J = F^\times$, W_F acts through the Artin map.

5.4. **Lubin-Tate tower.** Let $\mathbb{X}_0$ be a formal $\mathcal{O}_F$-module with height h, denoted as $\mathrm{ht}(\mathbb{X}_0) = h$ over $\overline{\mathbb{F}}$, which means that the kernel of multiplication by ϖ_F is a finite group scheme of rank q^h over $\overline{\mathbb{F}}$. Let $\mathcal{C}$ be the category of complete noetherian local $\mathcal{O}_{\breve{F}}$-algebra with residue field $\overline{\mathbb{F}}$.

Definition 5.4. For $A \in \mathcal{C}$, *deformations of* $\mathbb{X}_0$ *over* A are pairs (X, ι) such that X is a 1-dimensional formal $\mathcal{O}_F$-module over A, with an isomorphism $\iota : \mathbb{X}_0 \xrightarrow{\sim} X_{\mathbb{F}} = X \otimes_A \overline{\mathbb{F}}$.

Definition 5.5. For $A \in \mathcal{C}$ with maximal ideal $\mathfrak{m}_A$, we define *a structure of level* m on a deformation $(X, \iota) \in M_0(A)$ as an A-module homomorphism

$$\alpha : (\mathcal{O}_F/\varpi^m\mathcal{O}_F)^h \to X[\varpi^m](A) \subset \mathfrak{m}_A$$

such that $[\varpi]_X(T)$ is divisible by

$$\prod_{a \in (\mathcal{O}_F/\varpi^m\mathcal{O}_F)^h} (T - \alpha(a))$$

where $X[\varpi^m](A) := \{x \in A,\ [\varpi^m]_X(x) = 0\}$.

For each $m \geqslant 1$, let $U_m = 1 + \varpi_F^m(\mathrm{Mat}_{h\times h}(\mathcal{O}_F))$, which is called *the m-th principal congruence subgroup inside* $U_0 := \mathrm{GL}_h(\mathcal{O}_F)$.

Definition-Proposition 5.6. Define the set-valued functor $M^0_{U_m}$ on the category $\mathcal{C}$

$$\begin{aligned} &\mathcal{C} \to (\mathrm{Sets}) \\ &A \mapsto \{(X, \iota, \alpha),\ |\ (X, \iota) \text{ is a deformation of } \mathbb{X}_0 \text{ over } A, \alpha \text{ is a structure of level } m\} / \cong \end{aligned} \tag{5.2}$$

where $(X, \iota, \alpha) \cong (X', \iota', \alpha')$ if and only if there is an isomorphism $(X, \iota) \to (X', \iota')$ of formal $\mathcal{O}_F$-modules over A that is compatible with level structures, denoting such equivariant classes by $[X, \iota, \alpha]$. Then $M^{(0)}_{U_m}$ is represented by a $R^{(0)}_m \in \mathcal{C}$ which is finite flat over $\mathcal{O}_{\breve{F}}[[T_1, \ldots, T_{h-1}]]$ and $R^{(0)}_0 \cong \mathcal{O}_{\breve{F}}[[T_1, \ldots, T_{h-1}]]$ (non-canonically).

Proof. [Dri] Proposition 4.3. □

Remark 7. There is a universal formal group X^{univ} over $A^{\mathrm{univ}} := \mathcal{O}_{\breve{F}}[[T_1, \ldots, T_{h-1}]]$, together with an isomorphism $\iota^{\mathrm{univ}} : \mathbb{X}_0 \xrightarrow{\sim} X \otimes_{A^{\mathrm{univ}}} \overline{\mathbb{F}}$. Whenever A belongs to $\mathcal{C}$ and (X, ι) is a pair as above, there are unique elements (the maximal ideal of A) such that (X, ι) is the pull-back of $(X^{\mathrm{univ}}, \iota^{\mathrm{univ}})$ along the continuous map $\mathcal{O}_{\breve{F}}[[T_1, \ldots, T_{h-1}]] \to A$ sending T_i to x_i.

Let X be a formal $\mathcal{O}_F$-module over $A \in \mathcal{C}$, X is called to be of height h if $X_{\overline{\mathbb{F}}}$ has height h. For any isogeny $\iota_0 \in \mathrm{Hom}_{\mathcal{O}_F}(\mathbb{X}_0, X_{\overline{\mathbb{F}}})$, its height $\mathrm{ht}(\iota_0)$ is the number h such that rank of $\mathrm{Ker}(\iota_0)$ over $\overline{\mathbb{F}}$ is q^h. For any quasi-isogeny $\iota \in \mathrm{Hom}_{\mathcal{O}_F}(\mathbb{X}_0, X_{\overline{\mathbb{F}}}) \otimes_{\mathcal{O}_F} F$, we define its *height* by $\mathrm{ht}(\iota) = \mathrm{ht}(\varpi^r) - hr$, where r is chosen such that $\varpi^r\iota \in \mathrm{Hom}_{\mathcal{O}_F}(\mathbb{X}_0, X_{\overline{\mathbb{F}}})$.

Definition 5.7. Define the functor $M^{(k)}_{U_m}$ for $k \in \mathbb{Z}$ from $\mathcal{C}$ to category of sets:

$$\left\{ (X, \iota, \alpha), \begin{array}{l} \diamond\ X \text{ is a formal } \mathcal{O}_F\text{-module of height } h \text{ over } A, \\ \diamond \text{ A quasi-isogeny of height } k,\ \iota : \mathbb{X}_0 \to X_{\overline{\mathbb{F}}}, \\ \diamond \text{ A level } m\text{-structure } \alpha \end{array} \right\} / \cong \tag{5.3}$$

By the uniqueness of $\mathbb{X}_0$ up to isomorphism, we have $M^{(0)}_{U_m} \cong M^{(k)}_{U_m}$, but the isomorphism is not canonical. Therefore $M^{(k)}_{U_m}$ is also representable by a local $\mathcal{O}_{\breve{F}}$-algebra, denoted by $R^{(k)}_m$ which for varying k, m fixed, are all isomorphic. We set $M_{U_m} = \coprod_{k\in\mathbb{Z}} M^{(k)}_{U_m}$. Consider on $R^{(k)}_m$ the adic topology given by its maximal ideal $\mathfrak{m}_{R_m^{(k)}}$, we define the formal spectrum $\mathrm{Spf}(R_m^{(k)})$, also denoted by $M_{U_m}^{(k)}$, and denote their union by $M_{U_m} = \coprod_{k\in\mathbb{Z}} \mathrm{Spf}(R_m^{(k)})$.

Now we define group actions on M_{U_m}. Let $[X, \iota, \alpha]$ be any element of $M_{U_m}(A)$.

Denote by J or $D^\times$ the group of self quasi-isogenies of $\mathbb{X}_0$. Let $\mathcal{O}_D := \mathrm{End}_{\mathcal{O}_F}(\mathbb{X}_0)$ be the maximal order of central division algebra D with invariant $1/h$ over F. We define that $d \in D^\times$ acts on the right by $[X, \iota, \alpha].d = [X, \iota \circ d, \alpha]$. If $[X, \iota, \alpha] \in M_{U_m}^{(k)}(A)$, then $[X, \iota, \alpha].d \in M_{U_m}^{(k+v(\mathrm{Nrd}(b)))}(A)$, where $\mathrm{Nrd} : J \to F$ denotes the reduced norm.

Next, we define the action of the group $\mathrm{GL}_h(F)$ on the tower $\{M_{U_m}\}_{m\in\mathbb{Z}}$. Let $g \in \mathrm{Mat}_{h\times h}(\mathcal{O}_F)$, for integers $m \geqslant m' \geqslant 0$ such that

$$g\mathcal{O}_F^n \subset \varpi^{-(m-m')}\mathcal{O}_F^n$$

we define a right action $g_{m,m'} : M_{U_m}(A) \to M_{U_{m'}}(A)$ by $[X, \iota, \alpha].g$. By our assumption on g, we get $g\mathcal{O}_F^h \supset \mathcal{O}_F^h$, and $g\mathcal{O}_F^h/\mathcal{O}_F^h$ can be viewed as a subgroup of $(\mathcal{O}_F/\varpi^{-m}\mathcal{O}_F)^h$. A formal $\mathcal{O}_F$-module X' over A is defined by taking the quotient of X by the finite subgroup $\alpha(g\mathcal{O}_F^h/\mathcal{O}_F^h)$ denoted by $X' = X/\alpha(g\mathcal{O}_F^h/\mathcal{O}_F^h)$. Moreover, left multiplication by g induces an injection:

$$\varpi^{-m'}\mathcal{O}_F^h/\mathcal{O}_F^h \xrightarrow{g} \varpi^{-m}\mathcal{O}_F^h/g\mathcal{O}_F^h$$

and the composition with the map induced from level structure

$$(\varpi^{-m}\mathcal{O}_F^h/\mathcal{O}_F^h)/(g\mathcal{O}_F^h/\mathcal{O}_F^h) \to X/\alpha(g\mathcal{O}_F^h/\mathcal{O}_F^h) = X'$$

together gives a level m'-structure

$$\alpha' : \varpi^{-m'}\mathcal{O}_F^h/\mathcal{O}_F^h \to X'[\varpi^{m'}](A)$$

Finally, define ι' to be the composition of ι with $X_{\overline{\mathbb{F}}} \to (X')_{\overline{\mathbb{F}}}$. If element $[X, \iota, \alpha]$ is in $M_{U_m}^{(k)}(A)$, then $[X, \iota, \alpha].g$ is in $M_{U_{m'}}^{(k-v(\det g))}(A)$.

Now we define the action of a general $g \in \mathrm{GL}_h(F)$. We choose $r \in \mathbb{Z}$ such that $(\varpi^{-r}g)^{-1} \in \mathrm{Mat}_{h\times h}(\mathcal{O}_F)$. Then $m \geqslant m' \geqslant 0$ with

$$\varpi^{-r}g\mathcal{O}_F^n \subset \varpi^{-(m-m')}\mathcal{O}_F^n$$

and for $[X, \iota, \alpha] \in M_{U_m}(A)$ define $[X', \iota', \alpha'] = [X, \iota, \alpha].(\varpi^{-r}g)$ as above and put

$$[X, \iota, \alpha].g = [X', \iota' \circ \varpi^{-r}, \alpha']$$

This construction gives natural transformation

$$g : M_{U_m} \to M_{U_{m'}} \tag{5.4}$$

We can check that it is independent of r or ϖ. In particular, for each m there is an action of U_0 on M_{U_m} which commutes with action of $D^\times$.

5.5. **Rigid fibers.** We follow [Hub] §4 and define an adic space $t(M_{U_m}^{(k)})$ associated to $M_{U_m}^{(k)}$ (and to M_{U_m}). The set of points of the underlying topological space consists of all (equivalence classes of) continuous valuations $|\cdot|_v$ on $R_m^{(k)}$ such that $|f|_v \leqslant 1$ for all $f \in R_m^{(k)}$. The set of valuations $|\cdot|_v$ with $|\varpi|_v = 0$ is a closed subset which we denote by $V(\varpi)$. The open complement inherits the structure of an adic space and we put

$$\mathcal{M}_{U_m}^{(k)} = t(M_{U_m}^{(k)}) - V(\varpi), \text{ and } \mathcal{M}_{U_m} = \coprod_{k\in\mathbb{Z}} \mathcal{M}_{U_m}^{(k)}$$

For $m \geqslant m'$, there are canonical maps given by restriction of level structures

$$\mathcal{M}_{U_m} \to M_{U'_m}$$

which are étale and Galois with Galois group $U_{m'}/U_m \cong \mathrm{GL}_h(\mathcal{O}_F/\varpi^m)$.

For an open subgroup $U \subset U_0$, we choose m larger enough such that $U_m \subset U$. Since the action of U/U_m on $\mathcal{M}_{U_m}$ respects the components $\mathcal{M}^{(k)}_{U_m}$, we define

$$\mathcal{M}_U = \mathcal{M}_{U_m}/(U/U_m)$$

For any $g \in \mathrm{GL}_h(F)$ and an open subgroup $U \subset U_0$ such that $g^{-1}Ug \subset U_0$ there is a morphism of analytic spaces

$$\mathcal{M}_U \to \mathcal{M}_{g^{-1}Ug} \tag{5.5}$$

All the spaces $\mathcal{M}_U$ also come with an induced action of J which commutes with the morphisms induced by elements $g \in \mathrm{GL}_h(F)$.

We can give a description of what will happen when passing to limit, following [Wei] Lecture 2:

If we fix $m' = 0$, we have a tower of rigid spaces $\{\mathcal{M}_{U_m}\}_{m \geqslant 0}$, $\mathcal{M}_{U_m} \to \mathcal{M}_{U_0}$ is an étale covering of the rigid disk with Galois group $\mathrm{GL}_h(\mathcal{O}_F/\varpi^m)$. We would like to produce a space $\mathcal{M}$ which is the inverse limit of the $\mathcal{M}_{U_m}$. Such a space doesn't exist in the category of rigid spaces, so we shall have to content ourselves with the following definition on the level of points. Let $K' \supset W(\overline{\mathbb{F}})$ be a field admitting a valuation extending from $W(\overline{\mathbb{F}})$, then $\mathcal{M}(K')$ is the set of triples (X, ι, α) where

(i) X is a formal group over $\mathcal{O}_{K'}$,
(ii) A quasi-isogeny $\iota : \mathbb{X}_0 \otimes_{\overline{\mathbb{F}}} \mathcal{O}_{K'}/\varpi_{K'}\mathcal{O}_{K'} \to X \otimes_{\mathcal{O}_{K'}} \mathcal{O}_{K'}/\varpi_{K'}\mathcal{O}_{K'}$,
(iii) An isomorphism of K'-vector spaces $\alpha : \varprojlim_m (\mathcal{O}_{K'}/\varpi^m_{K'}\mathcal{O}_{K'})^h \otimes_{\mathcal{O}_{K'}} K' \xrightarrow{\sim} \varprojlim_m X[\varpi^m_{K'}] \otimes_{\mathcal{O}_{K'}} K'$

two points (X, ι, α), (X', ι', α') determine the same point of $\mathcal{M}$ if there is an isogeny $X \to X'$ translating one set of structure to the other.

We can define three group actions on $\{\mathcal{M}_{U_m}\}_{m \geqslant 0}$.

Denote by J or $D^\times$ as before. The element $d \in D^\times$ acts on $\mathcal{M}$ on the right by $(X, \iota, \alpha) \overset{d}{\mapsto} (X, \iota \circ d, \alpha)$.

Let $G = \mathrm{GL}_h(F)$, then each element $g \in G$ sends (X, ι, α) to $(X, \iota, \alpha \circ g)$.

There is also a Weil group W_F action on $\mathcal{M}$. Let $w \in W_F$ that induces Frob_q^n, $n \in \mathbb{Z}$ action on $\overline{\mathbb{F}}/\mathbb{F}$, where $\mathrm{Frob}_q : x \mapsto x^q$. Let (X, ι, α) be a $\mathbb{C}_p$-point of $\mathcal{M}$. Then we have the p-divisible group $w(X)$ by applying w coefficient-wise to a formal group law representing $\mathbb{X}$. And w acts on ι gives $\iota^w : \mathbb{X}_0^{q^n} \otimes \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p} \to w(X) \otimes \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}$, where $\mathbb{X}_0^{q^n}$ is $\mathbb{X}_0$ with coefficients translated after n times Frobenius map. There is also a canonical action on α.

5.6. **ℓ-adic cohomology groups.** Now we introduce the cohomology groups, from now on, we fix a prime number $\ell \neq p$, where $p = \mathrm{char}(\mathbb{F})$.

Lemma 5.8. *For any open subgroup $U \subset U_0$ and any $j \in \mathbb{Z}$, the $\mathbb{Q}_\ell$-vector spaces*

$$H^i_c(\mathcal{M}^{(j)}_U \times_{\breve{F}} \widehat{\overline{F}}, \mathbb{Q}_\ell) := \left(\varprojlim_r H^i_c(\mathcal{M}^{(j)}_U \times_{\breve{F}} \widehat{\overline{F}}, \mathbb{Z}/\ell^r\mathbb{Z})\right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \tag{5.6}$$

are finite-dimensional and the induced action of $\mathcal{O}_D^\times$ on these spaces is smooth. The cohomology groups $H^i_c(\mathcal{M}^{(j)}_U \times_{\breve{F}} \widehat{\overline{F}}, \mathbb{Q}_\ell)$ vanish for $i < h-1$ and $i > 2(h-1)$.

Proof. [Str2] Lemma 2.5.1. □

Next, we set

$$H_c^i(\mathcal{M}_U) = H_c^i(\mathcal{M}_U \times_{\breve{F}} \widehat{\overline{F}}, \overline{\mathbb{Q}}_\ell) := \bigoplus_{j\in\mathbb{Z}} H_c^i(\mathcal{M}_U^{(j)} \times_{\breve{F}} \widehat{\overline{F}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell \tag{5.7}$$

On each $\overline{\mathbb{Q}}_\ell$-vector space, using (5.5) there is an induced $U_0 \times J$ action as long as $g^{-1}Ug \subset U_0$

$$H_c^i(\mathcal{M}_{g^{-1}Ug}) \to H_c^i(\mathcal{M}_U)$$

These give rise to a representation of $\mathrm{GL}_h(F) \times J$ on

$$H_c^i(\mathcal{M}) := \varinjlim_U H_c^i(\mathcal{M}_U)$$

where limit is taken over all compact open subgroups $U \subset U_0$.

5.7. **Weak Kottwitz conjecture for** GL_h**,** $h \geqslant 2$**.** We continue to use the notations from the four subsections above, and give the statement of Kottwitz conjecture for GL_h, $h \geqslant 2$:

Theorem 5.9. *Let* JL *denote the local Jacquet-Langlands correspondence map* [HB] *§13, and recall that* LLC *denotes the local Langlands correspondence,* ibid. *§8. There exists a bijection between irreducible supercuspidal representations* ρ *of* $\mathrm{GL}_h(F)$ *and irreducible* h*-dimensional representations of* W_F *having the property that for all irreducible supercuspidal representations* ρ *with* $\overline{\mathbb{Q}}_\ell$*-coefficients we have:*

$$\begin{aligned} \mathrm{Hom}_G(H_c^{h-1}(\mathcal{M}, \overline{\mathbb{Q}}_\ell), \rho) &= \mathrm{JL}(\rho) \otimes \mathrm{LLC}(\rho). \\ \mathrm{Hom}_G(H_c^{k}(\mathcal{M}, \overline{\mathbb{Q}}_\ell), \rho) &= 0, \ k \neq h-1. \end{aligned} \tag{5.8}$$

as (virtual) representations of $J \times W_F$.

This result has been be stated in different forms by different authors, up to twist or taking contragredient of $\mathrm{LLC}(\rho)$. Here we give a sketch of the proof for the weak version without Weil group action. It is origianlly given by [Str1] and [Str2].

We write G to mean GL_h, H_c^i to mean $H_c^i(\mathcal{M})$ for simplicity. We put

$$\mathrm{Hom}_G(H_c^*, \rho) := \sum_i (-1)^i \mathrm{Hom}_G(H_c^i, \rho).$$

Proposition 5.10. *For each* $i \in \mathbb{Z}$*, the representation* $\mathrm{Hom}_G(H_c^i, \rho)$ *of* J *is a finite-dimensional and smooth, and in the Grothendieck group of admissible representations of* J*, the following equality holds:*

$$\mathrm{Hom}_G(H_c^*, \rho) = h \cdot (-1)^{h-1} \mathrm{JL}(\rho). \tag{5.9}$$

Here we ignore the Weil group action, thus $\mathrm{LLC}(\rho)$ is actually an h-dimensional Weil representation only appear as a multiplicity of $\mathrm{JL}(\rho)$ by h in the Grothendieck group of J.

Before going to the proof, let us do some preparations first.

By the fundamental resuit of Bushnell-Kutzko in [BK], we know that ρ is induced from a (finite-dimensional) irreducible representation λ of some open subgroup U_ρ that contains and is compact modulo $Z(G)$ *cf.* [BK], Theorem 8.4.1..

We thus write

$$\rho = \text{c-Ind}_{U_\rho}^G(\lambda) = \mathrm{Ind}_{U_\rho}^G(\lambda).$$

By Frobenius Reciprocity [HB] §2.4

$$\mathrm{Hom}_G(H_c^*, \rho) = \mathrm{Hom}_{K_\pi}(H_c^*, \lambda).$$

Moreover, the character of ρ is a locally constant function on the set of regular elliptic elements (*i.e.* whose characteristic polynomial is separable and irreducible) of G. For an element $g \in G$, and some character χ_ρ of ρ, there is a formula of characters (see [He])

$$\chi_\rho(g) = \sum_{g' \in G/U_\rho,\ (g')^{-1}gg' \in U_\rho} \chi_\lambda((g')^{-1}gg') \tag{5.10}$$

For regular elliptic g the number of g' such that $(g')^{-1}gg' \in U_\rho$ is finite.

For ρ as above, the representation $\pi = \mathrm{JL}(\rho)$ is characterized by the following identity. Let $g \in G$ and $b \in J$ be regular elliptic elements with the same characteristic polynomial. Then the following character relation holds

$$\chi_\pi(b) = (-1)^{h-1} \cdot \chi_\rho(g) \tag{5.11}$$

It is proved in [Row] Theorem 5.8.

Proof. We first prove the finiteness of $\mathrm{Hom}_G(H_c^i, \rho)$ as representation of J.

The element $\mathrm{diag}(\varpi, \dots, \varpi)$ in $Z(G)$ acts as scalar on ρ, and we denote this scalar by c^h for $c \in \overline{\mathbb{Q}}_\ell$. Put $\zeta(g) = c^{-v(\det(g))}$. For any $v \in H_c^*$, $\varpi \cdot v$ means the action of ϖ on v as an element of G. Then:

$$\begin{aligned} \mathrm{Hom}_{U_\rho}(H_c^*, \lambda) &= \mathrm{Hom}_{U_\rho}(H_c^* \otimes \zeta, \lambda \otimes \zeta) \\ &= \mathrm{Hom}_{U_\rho}\left((H_c^* \otimes \zeta)/\left\langle v - c^{-n}\varpi \cdot v \mid v \in H_c^i \right\rangle, \lambda \otimes \zeta\right) \end{aligned} \tag{5.12}$$

For $b \in J$, ξ is a character given by $\xi(b) = c^{-c(\mathrm{Nrd}(b))}$. We can show

$$(H_c^* \otimes \xi)/\left\langle v - c^{-n}\varpi \cdot v \mid v \in H_c^i \right\rangle$$

as representation of $G \times J$ is isomorphic to the natural representation of $G \times J$ on

$$\left(\varinjlim_U H_c^*(\mathcal{M}_U/\varpi^{\mathbb{Z}})\right) \otimes \xi$$

where limit is taken over all compact open subgroups $U \subset U_0$.

The isomorphism is constructed as follows: for $v \in H_c^*(\mathcal{M}_U^{(j)}, \overline{\mathbb{Q}}_\ell)$, v is mapped to $c^{nk}\varpi^{-k} \cdot v \in H_c^*(\mathcal{M}_U^{(j_0)}, \overline{\mathbb{Q}}_\ell)$, where $j = j_0 + nk$ with $0 \leqslant j_0 < n$. This is actually a $G \times J$-isomorphism, hence we get the following identity of representations of J:

$$\mathrm{Hom}_G(H_c^*, \rho) = \mathrm{Hom}_{U_\rho}(H_c^*(\mathcal{M}_\infty/\varpi^{\mathbb{Z}}), \lambda \otimes \zeta) \otimes \zeta^\vee$$

where

$$H_c^*(\mathcal{M}_\infty/\varpi^{\mathbb{Z}}) := \varinjlim_K H_c^*(\mathcal{M}_K/\varpi^{\mathbb{Z}}) = \varinjlim_U H_c^*(\mathcal{M}_U/\varpi^{\mathbb{Z}} \times_{\breve{F}} \widehat{\overline{F}}, \overline{\mathbb{Q}}_\ell)$$

Taken as a representation of G, $H_c^*(\mathcal{M}_\infty/\varpi^{\mathbb{Z}})$ is admissible because for any $U \subset U_0$ that is normal in U_ρ and satisfies $\lambda|_U$ is a multiple of trivial representations of U, the U-invariant vector subspace is just the finite dimensional vector space $H_c^*(\mathcal{M}_K/\varpi^{\mathbb{Z}})$, by Lemma 5.8.

Therefore

$$\mathrm{Hom}_G(H_c^*, \rho) = \mathrm{Hom}_I(H_c^*(\mathcal{M}_U/\varpi^{\mathbb{Z}}), \lambda \otimes \zeta) \otimes \zeta^\vee \tag{5.13}$$

where $I = U_\rho/\varpi^{\mathbb{Z}}U$ is finite group. This expression of $\mathrm{Hom}_G(H_c^*, \rho)$ involves only finite-dimensional vector spaces. □

The next step is to compute the trace of elliptic element $b \in J$ on (5.13). We have

$$\mathrm{tr}(b|\mathrm{Hom}_G(H_c^*,\rho)) = \frac{\zeta(b)^{-1}}{\#I}\sum_{\gamma\in I}\mathrm{tr}\left((\gamma,b^{-1})|H_c^*(\mathcal{M}_U/\varpi^{\mathbb{Z}})\right)\cdot\chi_{\lambda\otimes\zeta}(\gamma^{-1}) \tag{5.14}$$

Now we want to use Lefschetz trace formula to replace the terms in the above sum by an expression involving the number of fixed points and some additional terms.

Proposition 5.11. *In the above setting, let* $\mathrm{Fix}_U(\gamma,b^{-1})$ *be the number (with multiplicity) of fixed points of* (γ,b^{-1}) *on* $(\mathcal{M}_U/\varpi^{\mathbb{Z}})(\widehat{\overline{F}})$*, and* $\beta_U(\gamma,b^{-1})$ *be the class function with respect to* γ *such that*

$$\sum_{\gamma\in I}\beta_U(\gamma,b^{-1})\cdot\chi_{\lambda\otimes\zeta}(\gamma^{-1}) = 0 \tag{5.15}$$

Then there is a trace formula of the following form

$$\mathrm{tr}\left((\rho,b^{-1})|H_c^*(\mathcal{M}_U/\varpi^{\mathbb{Z}})\right) = \mathrm{Fix}_U(\gamma,b^{-1}) + \beta_U(\gamma,b^{-1}).$$

We will not prove this formula. We note that the class function $\beta_U(\cdot,b^{-1})$ can be written as $\sum_\tau a_\tau\chi_\tau$ where τ runs over the set of equivariant classes of irreducible representation of I. Those τ with non-zero a_τ are call *the representations that occurs in the boundary.* The preceding formula (5.15) says "no representation that gives rise to a supercuspidal representation occurs in the boundary".

Therefore,

$$\mathrm{tr}(b|\mathrm{Hom}_G(H_c^*,\rho)) = \frac{\zeta(b)^{-1}}{\#I}\sum_{\gamma\in I}\mathrm{Fix}_U(\gamma,b^{-1})\cdot\chi_{\lambda\otimes\zeta}(\gamma^{-1})$$

Finally, we state the *Fixed point theorem*:

Theorem 5.12. *Let* g_b *be element as in (5.11), then*

$$\mathrm{Fix}_U(\gamma,b^{-1}) = h\cdot\#\{\bar{g}\in G/\varpi^{\mathbb{Z}}U \mid \bar{g}^{-1}g_b\bar{g} = \gamma^{-1}\}$$

The identity $\bar{g}^{-1}g_b\bar{g}=\gamma^{-1}$ *means that for some representative* g *of* $\bar{g}\in G/\varpi^{\mathbb{Z}}U$ *we have* $\bar{g}^{-1}g_b\bar{g}\in U_\rho$ *and the class of* $\bar{g}^{-1}g_b\bar{g}$ *in* I *is* γ^{-1}.

Proof. See [Str1] §4. □

Now we come back to prove Theorem 5.10.

Proof. Let $b\in J$ be regular elliptic. Then the preceding discussion gives

$$\begin{aligned}\mathrm{tr}\,(b|\mathrm{Hom}_G(H_c^*,\rho)) &= \frac{\zeta(b)^{-1}}{\#I}\sum_{\bar{g}\in G/\varpi^{\mathbb{Z}}U,\ \bar{g}^{-1}g_b\bar{g}\in I} h\cdot\chi_{\lambda\otimes\zeta}(\bar{g}^{-1}g_b\bar{g})\\ &= h\sum_{\bar{g}\in G/U_\rho,\ \bar{g}^{-1}g_b\bar{g}\in U_\rho}\chi_\lambda(\bar{g}^{-1}g_b\bar{g})\\ &= h\cdot\chi_\rho(g_b)\\ &= h(-1)^{h-1}\chi_{\mathrm{JL}(\rho)}(b).\end{aligned} \tag{5.16}$$

where the first equality follows from Fix point theorem, the second equality comes from $\zeta(\bar{g}^{-1}g_b\bar{g}) = c^{-v(\det(g_b))} = c^{-v(\det(b))} = \zeta(b)$, the third equality follows from (5.10) and the last equality is by (5.11). □

6. Towards perfectoid spaces and the Fargues-Fontaine curve

We can prove Kottwitz conjecture for GL_h with some specially chosen b and $\{\mu\}$ either using global methods like [HT], or only using local methods but prove weaker version like [Str1] and [Str2]. Some of the obstruction for the pure local proof lies in that the Weil group action is complicated, and the description of infinite level of rigid fiber is lacking. In subsection 5.5, we have some trouble to describe the rigid fiber after passing to limit—we only have a description on the level of points for some finite field extension. This problem is solved if we use the embedding into products of universal covers of p-divisible groups, P.Scholze and J.Weinstein show that Lubin-Tate Tower at infinity level is perfectoid in some sense, as is done by [SW1]. We are thus led to introduce the perfectoid space setting and the Fargues-Fontaine curve.

6.1. Adic spaces.

All rings are commutative.

Definition 6.1. A *Huber ring* is a Hausdorff topological ring A containing an open subring A_0 such that the topology on A_0 coincides with the I-adic topology for some finitely generated ideal $I \subset A_0$. We call A_0 *a ring of definition*, and (A, I) *a couple of definition.*

We let $A^\circ \subset A$ be the subring of power-bounded elements.

Example 6.2. For $A = \mathbb{Q}_p$ with p-adic topology, $A^\circ = \mathbb{Z}_p$. We can take $A_0 = \mathbb{Z}_p$ and $I = (p)$.

Definition 6.3. Given A, a ring of integral elements is an open and integrally closed subring $A^+ \subset A$ with $A^+ \subseteq A^\circ$. A Huber ring is called a *Tate ring* if it contains a topologically nilpotent unit, *i.e.* $x \in A$ such that $\lim_{n\to\infty} x^n = 0$ and we call such x a *pseudo-uniformizer*.

Now we declare that from now on, the letter Γ is used to denote a totally ordered abelian group, **not** the absolute Galois group any more.

Definition 6.4. Given a topological ring A, a *continuous valuation* on A is a function $|\cdot| : A \to \Gamma \bigcup \{0\}$ satisfying:

(1) $|ab| = |a||b|$ and $|a+b| \leqslant \max(|a|, |b|)$,
(2) $|0| = 0$ and $|1| = 1$,
(3) for all $\gamma \in \mathrm{Im}\,|\cdot|$, the subset $\{a \in A : |a| < \gamma\}$ is open in A.

We say $|\cdot|$ and $|\cdot|'$ are equivalent if the condition

$$|a| \leqslant |b| \Leftrightarrow |a|' \leqslant |b|' \text{ for all } a,\ b \in A$$

Definition 6.5. Given (A, A^+) we define the *adic spectrum* $\mathrm{Spa}(A, A^+)$ to be the set of equivalence classes of continuous valuations $|\cdot|$ on A such that $|a| \leqslant 1$ for all $a \in A^+$. For $x \in Spa(A, A^+)$ write $|\cdot|_x : A \to \Gamma \bigcup \{0\}$ for a choice of valuation representing the equivalence class. We define the topology on $\mathrm{Spa}(A, A^+)$ by considering all open subsets generated by

$$\{x \in \mathrm{Spa}(A, A^+) \mid |f|_x \leqslant |g|_x \neq 0\}$$

for some $f,\ g \in A$.

Theorem 6.6. $\mathrm{Spa}(A, A^+)$ *is a* spectral space*,* i.e. Spec *of some ring. In particular,* $\mathrm{Spa}(A, A^+)$ *is quasi-compact.*

Definition 6.7. Let $X = \mathrm{Spa}(A, A^+)$ and $s \in A$ be arbitrary. Let $T \subset A$ be any finite subset generating an open ideal in A. A *rational subset* of X is one of the form

$$U(\frac{T}{s}) = \{x \in X \mid |t|_x \leqslant |s|_x \neq 0 \text{ for all } t \in T\}$$

Rational subsets are open, quasi-compact, and stable under finite intersection, and they generate the topology on X.

Proposition 6.8. *If $U \subset X = \mathrm{Spa}(A, A^+)$ is a rational subset, then there exists a complete Huber pair (A_U, A_U^+) with a map $\varphi : (A, A^+) \to (A_U, A_U^+)$ such that $\mathrm{Spa}(A_U, A_U^+) \to X$ is a homeomorphism onto U, and such that φ is universal for maps from (A, A^+) to complete Huber pairs which factor over U on adic spectra.*

Example 6.9. $\mathrm{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$ has two points: a generic point η corresponding to the p-adic valuation, and a special point s which factors through the trivial valuation on $\mathbb{F}_p$.

Example 6.10. Let $(A, A^+) = (\mathbb{Q}_p\langle T\rangle, \mathbb{Z}_p\langle T\rangle)$. The adic spectrum is the closed unit disk over $\mathbb{Q}_p$. Then

$$U(\frac{\{T, p\}}{p}) = \{|T|_x \leqslant |p|_x \neq 0\}$$

is the subdisk of radius $1/p$.

The universal property implies that (A_U, A_U^+) is unique up to unique isomorphism. It also implies that whenever $U \subset V$ is an inclusion of rational subsets, one gets $(A_V, A_V^+) \to (A_U, A_U^+)$.

Definition 6.11. Given $X = \mathrm{Spa}(A, A^+)$ we defines the *structure presheaf* $\mathcal{O}_X$ by $\mathcal{O}_X(U) = \varprojlim_{W \text{ rational } \subset U} A_W$. The *integral structure sheaf* $\mathcal{O}_X^+$ follows similarly:

$$\mathcal{O}_X^+(U) = \varprojlim_{W \text{ rational } \subset U} A_W^+$$

These are presheaves of complete topological rings. For all $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring, and the valuation $|\cdot|_x$ extends to a valuation $\mathcal{O}_{X,x} \to \Gamma_x \bigcup \{0\}$ whose kernel is the maximal ideal of $\mathcal{O}_{X,x}$.

Definition 6.12. Let A be a topological ring, A call *perfectoid* if A is complete, and A° is a bounded subring of A, there exists a topologically nilpotent unit $\varpi \in A$ such that $\varpi^p | p$, and the Frobenius map, *i.e.* $\mathrm{Frob}_A : A \to A$, $a \mapsto a^p$, induces a surjective map

$$\Phi : A^\circ/\varpi \to A^\circ/\varpi^p$$

We call such ϖ a *perfectoid pseudo-uniformiser.*

Definition 6.13. A perfectoid field K is a complete non-archimedean field K of residue characteristic p, equipped with a non-discrete valuation of rank 1, such that the Frobenius map $\Phi : \mathcal{O}_K/p \to \mathcal{O}_K/p$ is surjective, where $\mathcal{O}_K \subset K$ is the subring of elements of norm $\leqslant 1$.

Example 6.14. The following are examples of perfectoid rings: $\mathbb{C}_p$, $\widehat{\mathbb{Q}_p(\zeta_{p^\infty})}$, $\mathbb{C}_p\langle T^{1/p^\infty}\rangle$.

We explain a little bit more, ζ_{p^n} denote a primitive p^n-th root of unity, and $\widehat{\mathbb{Q}_p(\zeta_{p^\infty})}$ denotes the field obtained by adding all such roots and then take p-adic completion.

Definition 6.15. A subset S of a topological ring A is bounded if for all open neighborhoods U of 0 there exists an open neighborhood V of 0 such that $VS \subset U$.

$\mathcal{O}_X$ is not always a sheaf.

Proposition 6.16. *The structure presheaf on* $\mathrm{Spa}(A, A^+)$ *is a sheaf ((A, A^+) is call* sheafy *Huber pairs) in each of the following situations:*

(1) A is discrete, e.g. *the case of schemes.*

(2) *A admits a Noetherian ring of definition,* e.g. *the case of formal schemes.*
(3) *A is Tate and strongly Noetherian,* i.e. *$A\langle X_1, \ldots, X_n\rangle$ is Noetherian for any n,* e.g. *the case of rigid analytic varieties.*
(4) *A is Tate and stably uniform,* i.e. *for all rational subsets $U \subset \operatorname{Spa}(A, A^+)$ the subring $A_U^\circ \subset A_U$ is bounded.*
(5) *A is perfectoid.*

6.2. **Perfectoid spaces.**

Definition 6.17. A *perfectoid space* is a space glued locally from the adic spectra of perfectoid Huber pairs.

Definition 6.18. Let A be a complete topological ring in which p is topologically nilpotent. The tilt of A is

$$A^\flat := \varprojlim_{x \mapsto x^p} A = \{x = (x_0, x_1, x_2, \ldots) \in A^{\mathbb{N}} \mid x_{i+1}^p = x_i \text{ for all } i \geqslant 0\}$$

equipped with inverse limit topology. The multiplication is defined by coordinate wise multiplication, and addition law

$$(x+y)_i := \lim_{n \to \infty} (x_{i+n} + y_{i+n})^{p^n}.$$

Example 6.19. For ring who does not have many p-th root, $A^\flat$ is not interesting, *e.g.* $\mathbb{Q}_p^\flat = \mathbb{F}_p$.

This is a canonical map $\# : A^\flat \to A$ by $x \mapsto x_0$.

Proposition 6.20. *If (A, A^+) is a perfectoid Huber pair, then $(A^\flat, A^{+\flat})$ is a perfectoid Huber pair in characteristic p. Moreover, there a canonical homeomorphism* $\operatorname{Spa}(A, A^+)$ *taking $|\cdot|_x \mapsto |\cdot|_x \circ \#$ which*
(1) *induces a bijection of rational subsets $U \xrightarrow{\sim} U^\flat$,*
(2) *induces an isomorphism of rings $\mathcal{O}_X(U) \xrightarrow{\sim} \mathcal{O}_{X^\flat}(U^\flat)$.*

This operation glues to a functor $X \mapsto X^\flat$ from perfectoid spaces to perfectoid spaces in characteristic p.

Proposition 6.21. *Given a perfectoid space X, tilting induces an equivalence of categories*

$$\{\textit{Perfectoid spaces } Y/X\} \xrightarrow{\sim} \left\{\textit{Perfectoid spaces } Y^\flat/X^\flat\right\}$$

Example 6.22. If $A = \mathbb{C}_p$, then $A^\flat \cong \widehat{\overline{\mathbb{F}_p((t))}}$.

6.3. **Analytic adic spaces.**

Definition 6.23. A point x in an adic space is *analytic* if there exists a rational neighborhood $U = \operatorname{Spa}(A, A^+)$ of x where A is Tate.

Let A be a complete Tate ring, ϖ a pseudo-uniformiser of A, A_0 is a ring of definition. Then we define a norm

$$|\cdot| : A \to \mathbb{R}_{>0}, \quad a \mapsto \inf_{n \in \mathbb{Z} : \varpi^n a \in A_0} 2^n.$$

This induces a topology on A. Therefore, Tate rings are Banach rings.

Now we introduce rank one generalization. If $x \in \operatorname{Spa}(A, A^+)$ corresponds to $|\cdot|_x : A \to \Gamma$, then $\gamma = |\varpi|_x = |\varpi(x)| \in \Gamma$ must satisfy $\gamma^n \to 0$ as $n \to \infty$. There exists a map $\Gamma \to \mathbb{R}$ sending $\gamma \mapsto \frac{1}{2}$. Then we define a new valuation

$$|\cdot|_{\tilde{x}} : A \xrightarrow{|\cdot|_x} \Gamma \to \mathbb{R}_{>0}.$$

The corresponding $\tilde{x} \in \operatorname{Spa}(A, A^+)$ is an $\mathbb{R}_{>0}$-valued point which specialize to x, *i.e.* $\tilde{x} \rightsquigarrow x$. Then $|\cdot|_{\tilde{x}} \leqslant |\cdot|$, the set of rank 1 points of $\operatorname{Spa}(A, A^+)$ coincides with the set of rank 1 valuations $\leqslant |\cdot|$.

The point $\tilde{x}$ does not depend on the choice of ϖ. If ϖ' is another choice of uniformizer, then

$$\frac{\log|\varpi(\tilde{x})|}{\log|\varpi'(\tilde{x})|} \in \mathbb{R}_{>0}.$$

Note if A is a Huber ring, we may abbreviate $\operatorname{Spa} A = \operatorname{Spa}(A, A^\circ)$.

Let C be an algebraically closed perfectoid field.

Example 6.24. Consider $\operatorname{Spa}(k[[t]]) \times_{\operatorname{Spa} k} \operatorname{Spa}(k[[u]]) = \operatorname{Spa} k[[t, u]]$. This contains a special point s such that $|t(s)| = |u(s)| = 0$ outside which at least one of u or t is non-vanishing. Both t and u are topologically nilpotent. So s is the only non-analytic point. The complement $\mathcal{Y}$ is covered by two rational subsets

$$\begin{aligned} U(|t| \leqslant |u| \neq 0) &= \operatorname{Spa}\left(k((u))\langle\frac{t}{u}\rangle, k[[u]]\langle\frac{t}{u}\rangle\right) \\ U(|u| \leqslant |t| \neq 0) &= \operatorname{Spa}\left(k((t))\langle\frac{u}{t}\rangle, k[[t]]\langle\frac{u}{t}\rangle\right). \end{aligned} \tag{6.1}$$

Given $x \in \mathcal{Y}$, let $\kappa(x) = \dfrac{\log|u(\tilde{x})|}{\log|t(\tilde{x})|} \in [0, \infty]$. This defines a continuous surjective $\kappa : \mathcal{Y} \to [0, \infty]$. We see $\operatorname{Spa} k((t)) \times_{\operatorname{Spa} k} \operatorname{Spa} k((u)) = \kappa^{-1}(0, \infty)$.

Let C be an algebraically closed perfectoid field containing $\mathbb{F}_p$, let $A_{\text{inf}} := W(\mathcal{O}_C)$ with $(p, [\varpi])$-adic topology, $0 < |\varpi| < 1$.

Definition 6.25. We define $\mathcal{Y}$ as $\operatorname{Spa}(A_{\text{inf}}, A_{\text{inf}}) \backslash \{s\}$ where s is the point such that $|\varpi(s)| = |p(s)| = 0$. Then $\mathcal{Y}$ is analytic, and there exists $\kappa : \mathcal{Y} \to [0, \infty]$ defined by $\kappa(x) = \dfrac{\log|p(\tilde{x})|}{\log|\varpi(\tilde{x})|} \in [0, \infty]$.

The Frobenius Φ_C acts on $\mathcal{Y}$, and $\kappa(\Phi_C(y)) = p\kappa(y)$. In particular, Φ_C acts discontinously on $\mathcal{Y}_{(0,\infty)} := \kappa^{-1}(0, \infty)$.

Let $I \subset (0, \infty)$ be a closed interval, $B_I := H^0(\kappa^{-1}(I)^\circ, \mathcal{O}_{\mathcal{Y}})$, where $\kappa^{-1}(I)$ means the interior of $\kappa^{-1}(I)$. It can be proved that B_I is strongly noetherian, $\mathcal{Y}_{(0,\infty)}$ is an adic space. Let $B := \varprojlim_I B_I$.

Definition 6.26. The *adic Fargues-Fontaine* curve is $\mathcal{X}_{(C)} := \mathcal{Y}_{(0,\infty)}/\Phi_C$.

6.4. **Untilts.**

Definition 6.27. An *untilt* of C to $\mathbb{Q}_p$ is a pair $(C^\#, i)$ where $C^\#/\mathbb{Q}_p$ is a perfectoid field, and $i : C \xrightarrow{\sim} C^{\#\flat}$ is an isomorphism.

We say that Fargues-Fontaine curve carries untilts of C:

Theorem 6.28. *There is a bijection between the untilts of C to $\mathbb{Q}_p$, modulo equivalence, to closed maximal ideals of B.*

Proof. Given a maximal ideal $\mathfrak{m}$, we have untilt $B/\mathfrak{m}$. Conversely, if $(C^\#, i)$ is an untilt, then we have a map of multiplicative monoids

$$\mathcal{O}_C \cong \varprojlim_{x \mapsto x^p} \to \mathcal{O}_{C^\#}$$

denoted by $x \mapsto x^\#$. This lifts to a ring homomorphism

$$W(\mathcal{O}_C) = A_{\text{inf}} \to \mathcal{O}_{C^\#}$$

sending $[f] \mapsto f^{\#}$. This extends to

$$W(\mathcal{O}_C)[1/p] \to C^{\#}.$$

There is a map $W(\mathcal{O}_C)[1/p] \to B_I$. For I large enough, this extend through B_I, and composing with $B \to B_I$ gives a homomorphism $B \to C^{\#}$. □

Recall that we have defined the adic Fargues-Fontaine curve $\mathcal{X} = \mathcal{Y}_{(0,\infty)}/\Phi_C$ and $B = H^0(\mathcal{Y}_{(0,\infty)}, \mathcal{O}_{\mathcal{Y}_{(0,\infty)}})$ which has an action of Φ_C. Let $B^{\Phi_C = p}$ denote the subset of elements of B consisting of those $b \in B$ such that $\Phi_C b = b^p$, we actually have a functor, as described in [FF] §10.2.1.

$$\{\text{Isocrystals}/k\} \to \{\text{Vector bundles}/\mathcal{X}\}$$

where $k = \overline{\mathbb{F}}_p$.

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