Chapter 5: An Introduction to Hilbert Spaces

Exercise 1. Consider the $\mathbb{C}$-vector space $\mathscr{C}^\circ([-1,1],\mathbb{C})$ endowed with the inner product $\langle,\rangle$ defined by

$$\langle f,g\rangle := \int_{-1}^{1} \overline{f(t)}g(t)dt.$$

Let

$$F = \left\{f \in \mathscr{C}^\circ([-1,1],\mathbb{C}) \mid f(t) = f(-t) \quad \forall t \in [-1,1]\right\}.$$

Compute $F^\perp$. Does one have $E = F \oplus F^\perp$?

Solution of exercise 1. Let $G := \left\{f \in \mathscr{C}^\circ([-1,1],\mathbb{C}) \mid f(t) = -f(-t) \quad \forall t \in [-1,1]\right\}$. All linear combinations of odd functions are odd, and all linear combinations of even functions are even; so F and G are linear subspaces. Suppose that $f \in F$ and $g \in G$. Then $\overline{f}g$ is odd, which gives $\langle f,g\rangle = \int_{-1}^{1}\overline{f(t)}g(t)dt = 0$. So $F \perp G$. And every $f \in \mathscr{C}^\circ([-1,1],\mathbb{C})$ can be written as

$$f = f_e + f_o, \quad f_e \in F, f_o \in G,$$

where

$$f_e(t) = \frac{1}{2}(f(t)+f(-t)), \quad f_o(t) = \frac{1}{2}(f(t)-f(-t)).$$

So $F \perp G$ and $\mathscr{C}^\circ([-1,1],\mathbb{C}) = F \oplus G$ is an orthogonal direct sum decomposition. □

Exercise 2. Consider the $\mathbb{K}$-vector space $\mathbb{K}[x]$, and endow it with the inner product

$$\langle P,Q\rangle = \sum_{n=0}^{+\infty} \frac{1}{n!^2}\overline{P^{(n)}(0)}Q^{(n)}(0).$$

1. Prove that $\langle,\rangle$ is a well defined inner product.
2. Prove that the subset

$$H = \{P \in \mathbb{K}[x]/P(1) = 0\}$$

is a hyperplane and that $H^\perp = \{0\}$.

Solution of exercise 2. Assume $\mathbb{K} = \mathbb{C}$.

1. Let $P(x) = \sum_{i=0}^{N} a_i x^i$, $Q(x) = \sum_{i=0}^{M} b_i x^i$ with a_i, $b_i \in \mathbb{C}$. Then $\langle P,Q\rangle = \sum_{i=0}^{\min(M,N)} \overline{a_i}b_i$. $\langle P,P\rangle = \sum_{i=0}^{N}|a_i|^2 \geq 0$. "=" if and only if $P = 0$. Sesquilinear and Hermitian symmetric are easy to check.
2. $H = \ker \ell$ where $\ell \in \mathbb{K}[x]^*\backslash\{0\}$ is the linear form defined by

$$\ell(P) := P(1).$$

It is easy to check it is a linear form. Let $P(x) = \sum_{i=0}^{N} a_i x^i$, $H = \{P \in \mathbb{K}[x] \mid \sum_i a_i = 0\}$. We have $x^i - x^{M+1} \in H$ for any $M \in \mathbb{N}$. For any element $Q(x) = \sum_{i=0}^{M} b_i x^i \in H^\perp$, the condition $\langle H^\perp, Q\rangle = 0$ implies $\langle x^i - x^{M+1}, Q\rangle = b_i = 0$ holds for any i. We conclude that $H^\perp = \{0\}$.

□

Exercise 3. Consider the function $f \in \mathscr{C}^\circ([0,2\pi],\mathbb{C})$ defined by

$$f(t) = \begin{cases} t^2 & \text{if} \quad t \in [0,\pi] \\ (t-2\pi)^2 & \text{if} \quad t \in [\pi, 2\pi]. \end{cases}$$

1. Compute the Fourier coefficients of f. (Hint: prove first that $c_n(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2 e^{-int} dt$.)
2. Apply Parseval's identity to compute the value of

$$\sum_{n=1}^{+\infty} \frac{1}{n^4}.$$

Solution of exercise 3.

1. We have

$$\begin{aligned} c_n(f) &= \frac{1}{2\pi}\langle e_n, f\rangle \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{-int} f(t)dt \\ &= \frac{1}{2\pi}\left(\int_0^{\pi} t^2 e^{-int} dt + \int_{\pi}^{2\pi} (t-2\pi)^2 e^{-int} dt\right) \\ \text{(Let } u = t-2\pi) \quad &= \frac{1}{2\pi}\left(\int_0^{\pi} t^2 e^{-int} dt + \int_{-\pi}^{0} u^2 e^{-inu} du\right) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2 e^{-int} dt \\ (-\frac{1}{in} d(e^{-int}) = e^{-int}dt, \& n \neq 0) \quad &= -\frac{1}{2\pi i n}\left(\int_{-\pi}^{\pi} t^2 d(e^{-int})\right) \\ \text{(Integration by parts)} \quad &= -\frac{1}{2\pi i n}\left(e^{-int} t^2 \,|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} e^{-int} 2t dt\right) \\ \text{(Integration by parts again)} \quad &= \ldots \\ &= \frac{2(-1)^n}{n^2}. \end{aligned} \tag{1}$$

For $n = 0$, $c_0(f) = \dfrac{\pi^2}{3}$.

2. By Theorem 2.16: [Parseval identity] For $f \in \mathscr{C}^\circ([0,2\pi],\mathbb{C})$, one has

$$\begin{aligned} \sum_{n=-\infty}^{+\infty} \left|c_n(f)\right|^2 &= \frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2 dt \\ \frac{\pi^4}{9} + 2\sum_{n=1}^{\infty}\left(\frac{2(-1)^n}{n^2}\right)^2 = \sum_{n=-\infty}^{\infty} \left|c_n(f)\right|^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^4 dx = \frac{\pi^4}{5} \end{aligned} \tag{2}$$

We have $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{8}\left(\frac{1}{5} - \frac{1}{9}\right) = \frac{\pi^4}{90}$.

□

Exercise 4. Consider the Hilbert space $\ell^2(\mathbb{N},\mathbb{C})$. Fix $N \in \mathbb{N}$ and consider the map

$$h : \begin{cases} \ell^2(\mathbb{N},\mathbb{C}) & \to & \mathbb{C} \\ (a_n)_{n\in\mathbb{N}} & \mapsto & \sum_{n=0}^N a_n. \end{cases}$$

1. Prove that h is a bounded linear functional on $\ell^2(\mathbb{N},\mathbb{C})$.
2. Prove that $H := \ker(h)$ is closed.
3. Prove that $\ell^2(\mathbb{N},\mathbb{C}) = H \oplus H^\perp$.
4. Compute $H^\perp$.
5. Compute p_H.

Solution of exercise 4. Please read subsection 3.1.1 of chapV of the lecture notes.

1. Recall/read Proposition 5.1 of chapV. $|h((a_n)_{n\in\mathbb{N}})| = |\sum_{i=0}^N a_i| = |\sum_{i=0}^N 1\cdot a_i| \le \sqrt{N+1}\|(a_n)_{n\in\mathbb{N}}\|$. (Cauchy-Schwarz inequality applied to $(1,\cdots,1,0,\dots)$ (1 at first $N+1$ places, 0 elsewhere) and $(a_n)_{n\in\mathbb{N}}$; $(a_n)_{n\in\mathbb{N}} \in \ell^2$ so $\|(a_n)_{n\in\mathbb{N}}\| < \infty$).
2. For any $\{x_n\} \subset H$ with $x_n \to x$.
$$h(x) = \lim_{n\to\infty} h(x_n) = 0$$
Thus $x \in H$ and the nullspace is closed.
3. For any $x \in \ell^2(\mathbb{N},\mathbb{C})$, we decompose x as $x = \left(x - \frac{h(x)}{h(y_0)}y_0\right) + \dfrac{h(x)}{h(y_0)}y_0$, for any $y_0 \notin H$. We see $x - \frac{h(x)}{h(y_0)}y_0 \in H$ and $\dfrac{h(x)}{h(y_0)}y_0 \in \operatorname{Span}\{y_0\}$, thus $\ell^2(\mathbb{N},\mathbb{C}) = H \bigoplus \operatorname{Span}\{y_0\}$ (this is true for any linear space). Pick up (wisely) $y_0 = (b_n)_{n\in\mathbb{N}} = (1,\cdots,1,0,\dots)$, for any $(a_n)_{n\in\mathbb{N}} \in H$, $\langle y_0, (a_n)_{n\in\mathbb{N}}\rangle = \langle (b_n)_{n\in\mathbb{N}}, (a_n)_{n\in\mathbb{N}}\rangle = h((a_n)_{n\in\mathbb{N}}) = 0$. We have $\operatorname{Span}\{y_0\} \subset H^\perp$, thus $\operatorname{Span}\{y_0\} = H^\perp$. In general:

   Theorem. *Let E be a Hilbert space, Y a closed subspace of E, $Y^\perp$ the orthogonal complement of Y.*

   (a) *$Y^\perp$ is a closed linear subspace of E;*
   (b) *Y and $Y^\perp$ are complementary subspaces, meaning that every x can be decomposed uniquely as a sum of a vector in Y and in $Y^\perp$;*
   (c) *$\left(Y^\perp\right)^\perp = Y$.*
4. Done.
5. For $y_0 = (1,\cdots,1,0,\dots)$, $p_H(x) = x - \frac{h(x)}{h(y_0)}y_0 = x - (\dfrac{h(x)}{N+1},\dots,\dfrac{h(x)}{N+1},0,\dots)$.

□

Exercise 5. Fix $N \in \mathbb{N}$. Consider the linear form $h : \ell^2(\mathbb{N},\mathbb{C}) \to \mathbb{C}$ defined by

$$h\Big((a_n)_{n\in\mathbb{N}}\Big) = a_N.$$

1. Prove that h is continuous.

2. Determine the unique element $u \in \ell^2(\mathbb{N},\mathbb{C})$ such that

$$\ell(v) = \langle u, v\rangle \quad \forall v \in \ell^2(\mathbb{N},\mathbb{C}).$$

Solution of exercise 5.

1. Set $u = (0,\ldots,0,1,0,\ldots)$ (1 at $N+1$-th place), $|h\left((a_n)_{n\in\mathbb{N}}\right)| = |\langle u,(a_n)_{n\in\mathbb{N}}\rangle| \leq \|(a_n)_{n\in\mathbb{N}}\|$ by Cauchy-Schwarz.
2. By Theorem 5.6, u is unique.

□

Exercise 6. Consider the $\mathbb{C}$-vector space $\mathscr{C}^\circ([0,1],\mathbb{C})$ endowed with the inner product $\langle,\rangle$ defined by

$$\langle f, g\rangle := \int_0^1 \overline{f(t)}g(t)dt.$$

Consider the element $\ell \in \mathscr{C}^\circ([0,1],\mathbb{C})^*$ defined by

$$\ell(f) = f(0)$$

Prove that ℓ is not continuous.

Solution of exercise 6. Set $H_2 = \mathrm{Ker}\,\ell$. We have seen in 1.3.2 Second example that $H_2 \oplus H_2^\perp \neq \mathscr{C}^0([0,2\pi],\mathbb{C})$. If ℓ is continuous, by the proof of exercise 4, we will have $H_2 \oplus H_2^\perp = \mathscr{C}^0([0,1],\mathbb{C})$. Contradiction! Or you construct ... □

Exercise 9. Consider the $\mathbb{R}$-vector space $\mathscr{C}^\circ([-1,1],\mathbb{R})$ endowed with the inner product $\langle,\rangle$ defined by

$$\langle f, g\rangle := \int_{-1}^1 f(t)g(t)dt.$$

For any $k \in \mathbb{N}$ set

$$L_n := \frac{1}{2^n n!}\left[\left(x^2-1\right)^n\right]^{(n)} \in \mathbb{R}[x].$$

This is the so-called n-th Legendre polynomial.

1. For any $n \in \mathbb{N}$. Determine the leading coefficient of L_n.
2. Prove that the family $(L_n)_{n\in\mathbb{N}}$ is an orthogonal family in $\mathscr{C}^\circ([-1,1],\mathbb{R})$.
3. For any $n \in \mathbb{N}$, compute $\|L_n\|$.
4. Prove that if one applies the Gram-Schmidt process to the family $(x^n)_{n\in\mathbb{N}}$, one obtains the family $\left(\frac{L_n}{\|L_n\|}\right)_{n\in\mathbb{N}}$.

Solution of exercise 7.

1. The leading coefficient is $\frac{(2n)!}{2^n(n!)^2}$ of x^n.

2. Show that if f is indefinitely differentiable on $[-1,1]$, then

$$\int_{-1}^{1} L_n(x)f(x)dx = (-1)^n \frac{1}{2^n n!} \int_{-1}^{1} \left(x^2-1\right)^n f^{(n)}(x)dx$$

(Integration by parts). In particular, L_n is orthogonal to x^m whenever $m < n$. Hence $\{L_n\}_{n=0}^{\infty}$ is an orthogonal family.

3.

$$\begin{aligned}
\|L_n\| &= (-1)^n \frac{1}{2^n n!} \int_{-1}^{1} \left(x^2-1\right)^n L_n^{(n)}(x)dx \\
&= (-1)^n \frac{1}{2^n n!} \int_{-1}^{1} \left(x^2-1\right)^n \frac{d^{2n}}{dx^{2n}}\left[\left(x^2-1\right)^n\right]dx \\
&= (-1)^n \frac{(2n)!}{2^{2n}(n!)^2} \int_{-1}^{1} \left(x^2-1\right)^n dx
\end{aligned} \tag{3}$$

Using gamma functions or Integration by parts n time to find this equals $\frac{2}{n+2}$.

4. We can prove that any polynomial of degree n that is orthogonal to $1, x, x^2, \ldots, x^{n-1}$ is a constant multiple of L_n by comparing dimensions. Thus $\{L_n\}$ is the family obtained by applying the Gram-Schmidt process to $1, x, x^2, \ldots, x^n, \ldots$.

□